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ON ALMOST (m,n)-QUASI-IDEALS OF SEMIGROUPS AND THEIR FUZZIFICATIONS

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Abstract. In this paper, we introduce the concepts of almost (m,n)-quasi-ideals and fuzzy almost (m,n)-quasi-ideals of semigroups. The basic properties of those ideals are investigated. Furthermore, we provide the relationships between almost (m,n)-quasi-ideals and their fuzzifications. Eventually, the minimality, prime and semiprime are examined in such relationships.

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1. Introduction and preliminaries

The notion of quasi-ideals of semigroups have been introduced and studied by Steinfeld [7] in 1956. He defined a quasi-ideal of a semigroup S to be a nonempty subset Q of S if whenever $SQ \cap QS \subseteq Q$. Afterwards, Ansari, Khan and Kaushik [1] gave the generalization of quasi-ideals namely (m,n)-quasi-ideals where m and n are positive integers. A nonempty subset Q of a semigroup S is an (m,n)-quasi-ideal of S if $S^mQ \cap QS^n \subseteq Q$. So that a quasi-ideal of a semigroup S is a (1,1)-quasi-ideal of S. Besides, the concept of (m,n)-quasi-ideals have also been examined in other structures such as in rings [3], in LA-semigroups [4], etc.

In 1965, Zadeh [9] first introduced the fundamental fuzzy set concept. Later, the fuzzy sets are applied in various fields, especially in mathematics. We say that f is a fuzzy subset of a set S if f is a function from S into the closed interval $[0,1]$. For any two fuzzy subsets f and g of S,

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1. $f \cap g$ is a fuzzy subset of S defined by

$$(f \cap g)(x) = \min\{f(x), g(x)\} \qquad \text{for all } x \in S.$$

2. $f \cup g$ is a fuzzy subset of S defined by

$$(f \cup g)(x) = \max\{f(x), g(x)\} \qquad \text{for all } x \in S.$$

3. $f \subseteq g$ if $f(x) \leq g(x)$ for all $x \in S$.

For a fuzzy subset f of S, the *support* of f is defined by

$$supp(f) = \{x \in S \mid f(x) \neq 0\}.$$

The *characteristic mapping* of a subset A of S is a fuzzy subset of S defined by

$$C_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the semigroup S can be considered as a fuzzy subset of itself because $S(x) = C_S(x) = 1$ for all $x \in S$. In 1980, Pu and Liu [6] gave the definition of fuzzy points. Let S be a nonempty set. For $x \in S$ and $\alpha \in (0,1]$, a *fuzzy point* x_α of S is a fuzzy subset of S defined by for any $y \in S$,

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

Let $F(S)$ be the set of all fuzzy subsets of a semigroup S. For each $f, g \in F(S)$, a fuzzy subset $f \circ g$ of S is called a *product* of f and g defined by for any $x \in S$,

$$(f \circ g)(x) = \begin{cases} \sup\limits_{x=ab} \{\min\{f(a), g(b)\}\} & \text{if } x = ab \text{ for some } a, b \in S, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $F(S)$ is a semigroup with the product $\circ$. An introductory definition of left, right, two-sided almost ideals of semigroups were launched in 1980 by Grosek and Satko [5]. In 1981, the concepts of almost ideals and bi-ideals of semigroups were used to define almost bi-ideals in semigroups by Bogdanovic [2]. Likewise, Wattanatripop, Chinram and Changphas examined some properties of almost quasi-ideals (quasi-A-ideals) of semigroups in [10]. Furthermore, they investigated the fuzzy almost ideals of semigroups and gave the relationships between almost ideals of semigroups and thier fuzzifications. Next, the almost (m,n)-ideals and fuzzy almost (m,n)-ideals of semigroups have been introduced and studied in the same manner by Suebsung, Wattanatripop and Chinram in [8].

The aim of this paper is to introduce the concepts of almost (m,n)-quasi-ideals and the concept of fuzzy almost (m,n)-quasi-ideals of semigroups. We provide their basic properties and the relationships between them. Eventually, the minimality, prime and semiprime are examined in those relationships.

2. Almost (m,n)-quasi-ideals

First of all, the definition of almost (m,n)-quasi-ideals of semigroups is presented by using the concepts of (m,n)-ideals in [1] and almost ideals in [16]. Throughout this paper, let $m,n \in \mathbb{N}$ where $\mathbb{N}$ denote the set of all positive integers.

Definition 2.1. *Let S be a semigroup. A nonempty subset A of S is called an almost (m,n)-quasi-ideal of S if*

$$A^m s \cap sA^n \cap A \neq \emptyset \text{ for all } s \in S.$$

Remark 2.1. *The following statements hold.*

(1) *An almost $(1,1)$-ideal of a semigroup S is an almost quasi-ideal of S which were defined in [15].*

(2) *Every (m,n)-quasi-ideal of a semigroup S is an almost (m,n)-quasi-ideal of S. This means that the almost (m,n)-quasi-ideals are the generalization of the (m,n)-quasi-ideals of semigroups.*

(3) *Consider the semigroup $\mathbb{Z}_6$ under usual addition. Then $A = \{\overline{1},\overline{4},\overline{5}\}$ is a $(1,1)$-quasi-ideal of $\mathbb{Z}_6$ but A is not a subsemigroup of $\mathbb{Z}_6$. Therefore, an almost (m,n)-quasi-ideal of a semigroup S need not be a subsemigroup of S and need not be an (m,n)-quasi-ideal of S.*

Proposition 2.1. *If A is an almost (m,n)-quasi-ideal of a semigroup S, then every subset H of S such that $A \subseteq H$ is an almost (m,n)-quasi-ideal of S.*

Proof. Assume that A is an almost (m,n)-quasi-ideal of S and H is a subset of S such that $A \subseteq H$. We obtain that $\emptyset \neq A^m s \cap sA^n \cap A \subseteq H^m s \cap sH^n \cap H$ for all $s \in S$. Therefore, H is an almost (m,n)-quasi-ideal of S. □

Corollary 2.1. *The union of two almost (m,n)-quasi-ideals of a semigroup S is an almost (m,n)-quasi-ideal of S.*

Proof. Let A_1 and A_2 be any two almost (m,n)-quasi-ideals of S. Since $A_1, A_2 \subseteq A_1 \cup A_2$, the set $A_1 \cup A_2$ is an almost (m,n)-quasi-ideal of S by Proposition 2.1. □

From Corollary 2.1, of course, it is easy to see that if A_1 or A_2 is an almost (m,n)-quasi-ideal of a semigroup S, then the result is still true. Nevertheless, if we consider in case of the intersection instead of the union, then the statement of Corollary 2.1 is not true as shown in the following example.

Example 2.1. *Consider the semigroup $\mathbb{Z}_6$ under the usual addition. We have $A_1 = \{\overline{1},\overline{4},\overline{5}\}$ and $A_2 = \{\overline{1},\overline{2},\overline{5}\}$ are almost (1,1)-quasi-ideals of $\mathbb{Z}_6$ but $A_1 \cap A_2 = \{\overline{1},\overline{5}\}$ is not an almost (1,1)-quasi-ideal of $\mathbb{Z}_6$.*

Theorem 2.1. *Let S be a semigroup such that $|S| > 1$. A semigroup S has no proper almost (m,n)-quasi-ideal if and only if for any $a \in S$ there exists $s_a \in S$ such that $(S \setminus \{a\})^m s_a \cap s_a(S \setminus \{a\})^n = \{a\}$.*

Proof. Assume that S has no proper almost (m,n)-quasi-ideal and let $a \in S$. Then $S \setminus \{a\}$ is not an almost (m,n)-quasi-ideal of S. This implies that there exists $s_a \in S$ such that $[(S \setminus \{a\})^m s_a \cap s_a(S \setminus \{a\})^n] \cap (S \setminus \{a\}) = \emptyset$. Therefore, we obtain that $(S \setminus \{a\})^m s_a \cap s_a(S \setminus \{a\})^n = \{a\}$.

Conversely, let $a \in S$. Thus there exists $s_a \in S$ such that $(S \setminus \{a\})^m s_a \cap s_a(S \setminus \{a\})^n = \{a\}$. This implies that

$$[(S \setminus \{a\})^m s_a \cap s_a(S \setminus \{a\})^n] \cap (S \setminus \{a\}) = \emptyset.$$

Hence, $S \setminus \{a\}$ is not an almost (m,n)-quasi-ideal of S for all $a \in S$. Suppose that S has a proper almost (m,n)-quasi-ideal B of S. Then $B \subseteq S \setminus \{a'\}$ for some $a' \in S$. So, we have that $S \setminus \{a'\}$ is also an almost (m,n)-quasi-ideal of S by Proposition 2.1. This is a contradiction. Therefore, S has no proper almost (m,n)-quasi-ideal. $\square$

Theorem 2.2. *Let S be a semigroup such that $|S| > 1$ and $a \in S$. If $S \setminus \{a\}$ is not an almost (m,n)-quasi-ideal of S, then at least one of the following statements is true.*

(1) $a = a^{m+1}$.
(2) $a = a^{m^3+1}$.
(3) $a = a^{m(m+1)+1}$.

Proof. Assume that $S \setminus \{a\}$ is not an almost (m,n)-quasi-ideal of S. By Theorem 2.1, there exists $s_a \in S$ such that $(S \setminus \{a\})^m s_a \cap s_a(S \setminus \{a\})^n = \{a\}$.

Case 1: $s_a \neq a$. Then $s_a \in S \setminus \{a\}$. We obtain that $(s_a)^m s_a = s_a(s_a)^n = a$. Suppose that $a \neq a^{m+1}$. This implies that $a^{m+1} \in S \setminus \{a\}$. Since $(a^{m+1})^m s_a = s_a(a^{m+1})^n$, we have that $(a^{m+1})^m s_a \in (S \setminus \{a\})^m s_a \cap s_a(S \setminus \{a\})^n$. Hence, $a = (a^{m+1})^m s_a$.

Case 1.1: If $a^{m^2} s_a = a$, then

$$a = (a^{m+1})^m s_a = a^{m+m^2} s_a = a^m a^{m^2} s_a = a^{m+1},$$

which is a contradiction.

Case 1.2: If $a^{m^2} s_a \neq a$, then $a^{m^2} s_a \in S \setminus \{a\}$. Thus

$$(a^{m^2} s_a)^m s_a = a.$$

This implies that

$$a = (a^{m^2} s_a)^m s_a = a^{m^3}(s_a)^m s_a = a^{m^3+1}.$$

Case 2: $s_a = a$. Suppose that $a \neq a^{m+1}$. Thus $a^{m+1} \in S \setminus \{a\}$. We have that $(a^{m+1})^m s_a = s_a(a^{m+1})^n = a$. Therefore,

$$a = (a^{m+1})^m a = a^{m(m+1)+1}.$$

$\square$

Corollary 2.2. *Let S be a semigroup such that $|S| > 1$ and $a \in S$. If $S \setminus \{a\}$ is not an almost quasi-ideal of S, then $a = a^2$ or $a = a^3$.*

Proof. The proof is completed by choosing $m = 1$ in Theorem 2.2. □

3. Fuzzy almost (m, n)-quasi-ideals

In this section, we define and investigate the fuzzy almost (m, n)-quasi-ideals and provide the relationships between fuzzy almost (m, n)-quasi-ideals and almost (m, n)-quasi-ideals. Let f be a fuzzy subset and x_α be a fuzzy point of a semigroup S. For $k \in \mathbb{N}$, let $f^k := \underbrace{f \circ f \circ \ldots \circ f}_{k \text{ times}}$.

Proposition 3.1. *Let f, g and h be fuzzy subsets of S.*

(1) *If $f \subseteq g$, then $f^n \subseteq g^n$ for all $n \in \mathbb{N} \cup \{0\}$.*
(2) *If $f \subseteq g$, then $f \circ h \subseteq g \circ h$ and $h \circ f \subseteq h \circ g$.*
(3) *If $f \subseteq g$, then $f \cap h \subseteq g \cap h$ and $h \cap f \subseteq h \cap g$.*

Proof. The proof is straightforward. □

Definition 3.1. *A fuzzy subset f of a semigroup S is called a fuzzy almost (m, n)-quasi-ideal of S if*

$$(f^m \circ x_\alpha) \cap (x_\alpha \circ f^n) \cap f \neq 0$$

for any fuzzy point x_α of S.

This implies that f is a fuzzy almost (m, n)-quasi-ideal of S if for all fuzzy point x_α of S, there exists $y \in S$ such that $[(f^m \circ x_\alpha) \cap (x_\alpha \circ f^n) \cap f](y) \neq 0$.

Proposition 3.2. *Let f be a fuzzy almost (m, n)-quasi-ideal of S and g be a fuzzy subset of S such that $f \subseteq g$. Then g is a fuzzy almost (m, n)-quasi-ideal of S.*

Proof. Assume that f is a fuzzy almost (m, n)-quasi-ideal of S and g is a fuzzy subset of S such that $f \subseteq g$. Let x_α be a fuzzy point of S. We obtain

$$0 \neq (f^m \circ x_\alpha) \cap (x_\alpha \circ f^n) \cap f \subseteq (g^m \circ x_\alpha) \cap (x_\alpha \circ g^n) \cap g.$$

Therefore, g is a fuzzy almost (m, n)-quasi-ideal of S. □

Corollary 3.1. *Let f and g be fuzzy almost (m, n)-quasi-ideals of S. Then $f \cup g$ is a fuzzy almost (m, n)-quasi-ideal of S.*

Proof. Since $f, g \subseteq f \cup g$ and by Proposition 3.2, we have $f \cup g$ is a fuzzy almost (m, n)-quasi-ideal of S. □

It is clear that the statement of Corollary 3.1 is still true if f or g is a fuzzy almost (m,n)-quasi-ideal of S.

Example 3.1. *Consider the semigroup $\mathbb{Z}_6$ under the usual addition. Define $f : \mathbb{Z}_6 \to [0,1]$ by*

$$f(\bar{0}) = 0, f(\bar{1}) = 0.2, f(\bar{2}) = 0, f(\bar{3}) = 0, f(\bar{4}) = 0.5, f(\bar{5}) = 0.3$$

and define $g : \mathbb{Z}_6 \to [0,1]$ by

$$g(\bar{0}) = 0, g(\bar{1}) = 0.8, g(\bar{2}) = 0.4, g(\bar{3}) = 0, g(\bar{4}) = 0, g(\bar{5}) = 0.3.$$

We have f and g are fuzzy almost (1,1)-quasi-ideals of $\mathbb{Z}_6$ but $f \cap g$ is not a fuzzy almost (1,1)-quasi-ideal of $\mathbb{Z}_6$.

Example 3.1 implies that, in general, the intersection of two fuzzy almost (m,n)-quasi-ideals of S need not be a fuzzy almost (m,n)-quasi-ideal of S.

Lemma 3.1. *Let A be a subset of S and $n \in \mathbb{N} \cup \{0\}$. Then $(C_A)^n = C_{A^n}$.*

Proof. The proof is straightforward. □

Theorem 3.1. *Let A be a nonempty subset of a semigroup S. Then A is an almost (m,n)-quasi-ideal of S if and only if C_A is a fuzzy almost (m,n)-quasi-ideal of S.*

Proof. Assume that A is an almost (m,n)-quasi-ideal of S. Then $A^m s \cap sA^n \cap A \neq \emptyset$ for all $s \in S$. Let $s \in S$ and $\alpha \in (0,1]$. Thus there exists $x \in A^m s \cap sA^n \cap A$. So

$$[(C_{A^m} \circ s_\alpha) \cap (s_\alpha \circ C_{A^n}) \cap C_A](x) \neq 0.$$

By Lemma 3.1, we obtain

$$[((C_A)^m \circ s_\alpha) \cap (s_\alpha \circ (C_A)^n) \cap C_A](x) \neq 0.$$

Hence, C_A is a fuzzy almost (m,n)-quasi-ideal of S.

Conversely, suppose that C_A is a fuzzy almost (m,n)-quasi-ideal of S. Let $s \in S$ and $\alpha \in (0,1]$. Thus

$$((C_A)^m \circ s_\alpha) \cap (s_\alpha \circ (C_A)^n) \cap C_A \neq 0.$$

Then there exists $x \in S$ such that

$$[((C_A)^m \circ s_\alpha) \cap (s_\alpha \circ (C_A)^n) \cap C_A](x) \neq 0.$$

By Lemma 3.1, we have

$$[(C_{A^m} \circ s_\alpha) \cap (s_\alpha \circ C_{A^n}) \cap C_A](x) \neq 0.$$

Therefore, $x \in A^m s \cap sA^n \cap A$. Eventually, $A^m s \cap sA^n \cap A \neq \emptyset$ and then A is an almost (m,n)-quasi-ideal of S. □

Theorem 3.2. *Let f be a fuzzy subset of S. Then f is a fuzzy almost (m,n)-quasi-ideal of S if and only if $supp(f)$ is an almost (m,n)-quasi-ideal of S.*

Proof. Assume that f is a fuzzy almost (m,n)-quasi-ideal of S. Let $x \in S$. Then for any $\alpha \in (0,1]$, we have

$$(f^m \circ x_\alpha) \cap (x_\alpha \circ f^n) \cap f \neq 0.$$

Thus there exists $y \in S$ such that

$$[(f^m \circ x_\alpha) \cap (x_\alpha \circ f^n) \cap f](y) \neq 0.$$

So $f(y) \neq 0$ and $y = a_1a_2 \cdots a_m x = x b_1 b_2 \cdots b_n$ for some $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n \in S$ such that

$$f(a_1) \neq 0, f(a_2) \neq 0, \ldots, f(a_m) \neq 0, f(b_1) \neq 0, f(b_2) \neq 0, \ldots, f(b_n) \neq 0.$$

This implies that $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n, y \in supp(f)$. Thus

$$[((C_{supp(f)})^m \circ x_\alpha) \cap (x_\alpha \circ (C_{supp(f)})^n)](y) \neq 0 \text{ and } C_{supp(f)}(y) \neq 0.$$

Hence,

$$[((C_{supp(f)})^m \circ x_\alpha) \cap (x_\alpha \circ (C_{supp(f)})^n) \cap C_{supp(f)}](y) \neq 0.$$

So, $C_{supp(f)}$ is a fuzzy almost (m,n)-quasi-ideal of S. By Theorem 3.1, $supp(f)$ is an almost (m,n)-quasi-ideal of S.

Conversely, assume that $supp(f)$ is an almost (m,n)-quasi-ideal of S. By Theorem 3.1, $C_{supp(f)}$ is a fuzzy almost (m,n)-quasi-ideal of S. Let x_α be a fuzzy point of S. Then

$$((C_{supp(f)})^m \circ x_\alpha) \cap (x_\alpha \circ (C_{supp(f)})^n) \cap C_{supp(f)} \neq 0.$$

Thus there exists $y \in S$ such that

$$[((C_{supp(f)})^m \circ x_\alpha) \cap (x_\alpha \circ (C_{supp(f)})^n) \cap C_{supp(f)}](y) \neq 0.$$

Hence,

$$[((C_{supp(f)})^m \circ x_\alpha) \cap (x_\alpha \circ (C_{supp(f)})^n)](y) \neq 0 \text{ and } C_{supp(f)}(y) \neq 0.$$

Then there exist $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n \in supp(f)$ and $y = a_1a_2 \cdots a_m x = x b_1 b_2 \cdots b_n$. We obtain that

$$f(y) \neq 0, f(a_1) \neq 0, f(a_2) \neq 0, \ldots, f(a_m) \neq 0, f(b_1) \neq 0, f(b_2) \neq 0, \ldots, f(b_n) \neq 0.$$

Therefore,

$$[(f^m \circ x_\alpha) \cap (x_\alpha \circ f^n)](y) \neq 0.$$

This implies that

$$[(f^m \circ x_\alpha) \cap (x_\alpha \circ f^n) \cap f](y) \neq 0.$$

Therefore, f is a fuzzy almost (m,n)-quasi-ideal of S. □

3.1. **Minimal almost (m,n)-quasi-ideals and minimal fuzzy almost (m,n)-quasi-ideals.** In this subsection, the relationship between minimal almost (m,n)-quasi-ideals and minimal fuzzy almost (m,n)-quasi-ideals is presented.

Definition 3.2. *A fuzzy almost (m,n)-quasi-ideal f is said to be* minimal *if for all nonzero fuzzy almost (m,n)-quasi-ideals g of S such that $g \subseteq f$, we have $supp(f) = supp(g)$.*

Theorem 3.3. *Let A be a nonempty subset of a semigroup S. Then A is a minimal almost (m,n)-quasi-ideal of S if and only if C_A is a minimal fuzzy almost (m,n)-quasi-ideal of S.*

Proof. Assume that A is a minimal almost (m,n)-quasi-ideal of S. By Theorem 3.1, C_A is a fuzzy almost (m,n)-quasi-ideal of S. Let f be a fuzzy almost (m,n)-quasi-ideal of S such that $f \subseteq C_A$. It follows that $supp(f) \subseteq supp(C_A) = A$. Hence, $supp(f)$ is an almost (m,n)-quasi-ideal of S by Theorem 3.2. Since A is minimal, $supp(f) = A = supp(C_A)$. Therefore, C_A is minimal.

Conversely, suppose that C_A is a minimal fuzzy almost (m,n)-quasi-ideal of S. Let B be an almost (m,n)-quasi-ideal of S such that $B \subseteq A$. Then C_B is a fuzzy almost (m,n)-quasi-ideal of S such that $C_B \subseteq C_A$. Thus $B = supp(C_B) = supp(C_A) = A$. Hence, A is minimal. □

Corollary 3.2. *A semigroup S has no proper almost (m,n)-quasi-ideal if and only if for all fuzzy almost (m,n)-quasi-ideal f of S, $supp(f) = S$.*

Proof. The proof of this corollary follows by Theorem 3.3. □

3.2. **Prime almost (m,n)-quasi-ideals and prime fuzzy almost (m,n)-quasi-ideals.** In this subsection, we focus on the relationship between prime almost (m,n)-quasi-ideals and prime fuzzy almost (m,n)-quasi-ideals.

Definition 3.3. *Let S be a semigroup.*

(1) *An almost (m,n)-quasi-ideal A of S is called* prime *if for all $x, y \in S$,*

$$xy \in A \text{ implies } x \in A \text{ or } y \in A.$$

(2) *A fuzzy almost (m,n)-quasi-ideal f of S is called* prime *if for all $x, y \in S$,*

$$f(xy) \leq \max\{f(x), f(y)\}.$$

Theorem 3.4. *Let A be a nonempty subset of a semigroup S. Then A is a prime almost (m,n)-quasi-ideal of S if and only if C_A is a prime fuzzy almost (m,n)-quasi-ideal of S.*

Proof. Assume that A is a prime almost (m,n)-quasi-ideal of S. By Theorem 3.1, we obtain that C_A is a fuzzy almost (m,n)-quasi-ideal of S. Let $x, y \in S$. We consider into two cases:
Case 1: $xy \in A$. Then $x \in A$ or $y \in A$. Hence, $C_A(xy) \leq 1 = \max\{C_A(x), C_A(y)\}$.

Case 2: $xy \notin A$. Thus $C_A(xy) = 0 \leq \max\{C_A(x), C_A(y)\}$.
Therefore, C_A is a prime fuzzy almost (m,n)-quasi-ideal of S.

Conversely, suppose that C_A is a prime fuzzy almost (m,n)-quasi-ideal of S. By Theorem 3.1, we have that A is an almost (m,n)-quasi-ideal of S. Let $x, y \in S$ be such that $xy \in A$. Then $C_A(xy) = 1$, and it follows that $C_A(xy) \leq \max\{C_A(x), C_A(y)\}$. Hence, we obtain $\max\{C_A(x), C_A(y)\} = 1$. Consequently, $x \in A$ or $y \in A$. Therefore, A is a prime almost (m,n)-quasi-ideal of S. □

3.3. **Semiprime almost (m,n)-quasi-ideals and semiprime fuzzy almost (m,n)-quasi-ideals.** In this subsection, the relationship between semiprime almost (m,n)-quasi-ideals and semiprime fuzzy almost (m,n)-quasi-ideals is given.

Definition 3.4. *Let S be a semigroup.*

(1) *An almost (m,n)-quasi-ideal A of S is said to be* semiprime *if for all $x \in S$,*

$$x^2 \in A \textit{ implies } x \in A.$$

(2) *A fuzzy almost (m,n)-quasi-ideal f of S is said to be* semiprime *if for all $x \in S$,*

$$f(x^2) \leq f(x).$$

From Definitions 3.3 and 3.4, we see that every prime almost (m,n)-quasi-ideal of semigroups is semiprime almost (m,n)-quasi-ideal, and every prime fuzzy almost (m,n)-quasi-ideal of semigroups is semiprime fuzzy almost (m,n)-quasi-ideal.

Theorem 3.5. *Let A be a nonempty subset of a semigroup S. Then A is a semiprime almost (m,n)-quasi-ideal of S if and only if C_A is a semiprime fuzzy almost (m,n)-quasi-ideal of S.*

Proof. Assume that A is a semiprime almost (m,n)-quasi-ideal of S. We gain that C_A is a fuzzy almost (m,n)-quasi-ideal of S by Theorem 3.1. Let $x \in S$. We consider into two cases:
Case 1: $x^2 \in A$. Then $x \in A$. Thus $C_A(x^2) \leq 1 = C_A(x)$.
Case 2: $x^2 \notin A$. Thus $C_A(x^2) = 0 \leq C_A(x)$.
Therefore, C_A is a semiprime fuzzy almost (m,n)-quasi-ideal of S.

Conversely, suppose that C_A is a semiprime fuzzy almost (m,n)-quasi-ideal of S. By Theorem 3.1, A is an almost (m,n)-quasi-ideal of S. Let $x \in S$ be such that $x^2 \in A$. Then $C_A(x^2) = 1$. By assumption, we have $C_A(x^2) \leq C_A(x)$. It follows that $C_A(x) = 1$, so $x \in A$. Therefore, A is a semiprime almost (m,n)-quasi-ideal of S. □

4. Conclusions

In this paper, we have introduced the concepts of almost (m,n)-quasi-ideals and the concept of fuzzy almost (m,n)-quasi-ideals of semigroups. As the results, we obtained that those ideals preserve the basic properties in semigroups. In addition, we found that if A is a nonempty subset of a semigroup S, then A is a (minimal/prime/semiprime) almost (m,n)-quasi-ideal of S if and only if C_A is a (minimal/prime/semiprime) fuzzy almost (m,n)-quasi-ideal of S.

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