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On Diophantine equation $n^x + (9p)^y = z^2$

Theerapon Buatong[1], **Ronnason Chinram**[2]

[1]Mathematics Program
Faculty of Science and Technology
Songkhla Rajabhat University
Songkhla 90000, Thailand

[2]Division of Computational Science
Faculty of Science
Prince of Songkla University
Hat Yai, Songkhla 90110, Thailand

email: theerapon.bu@skru.ac.th, ronnason.c@psu.ac.th

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Abstract

In this paper, we consider integer solutions (n, p, x, y, z) for the Diophantine equation $n^x + (9p)^y = z^2$ where n is positive, p is prime and x, y, z are non-negative. We show that this equation has no solution if $n \equiv 1 \pmod 4$ and $p \equiv 1 \pmod 4$. Moreover, we find solutions of the Diophantine equation $n^x + (9p)^y = z^2$ in some cases.

1 Introduction and Preliminaries

Catalan's conjecture was conjectured by Catalan [2] in 1844 and proven by Mihăilescu [3] in 2002.

Theorem 1.1. (Mihăilescu's Theorem) *For any two integers $a, b > 1$, the Diophantine equation $x^a - y^b = 1$ has no solutions positive integers x and y, other than $(a, b, x, y) = (2, 3, 3, 2)$.*

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During 2013-2014, some solutions of the Diophantine equation $p^x+q^y = z^2$ were discovered [4, 5]. Later, Ardsalee and Manyuen [1] found a non-existence condition for solutions of the Diophantine equation $a^x + b^y = z^2$. In 2024, the Diophantine equation $n^x + (5p)^y = z^2$, where n, p, x, y and z are integers such that $n > 0$, p is prime, and x, y and z are non-negative, was considered in [6].

In this paper, we consider integer solutions n, p, x, y and z for the Diophantine equation $n^x + (9p)^y = z^2$ where n is positive, p is prime and x, y, z are non-negative. We provide a condition for which this Diophantine equation does not have a solution. On the other hand, we find some solutions of this Diophantine equation.

2 Main results

Theorem 2.1. *If* $n \equiv 1 \pmod 4$ *and* $p \equiv 1 \pmod 4$, *then the Diophantine equation* $n^x + (9p)^y = z^2$ *has no solution.*

Proof. Suppose, to get a contradiction, that the Diophantine equation $n^x + (9p)^y = z^2$ has a solution (n, p, x, y, z). By assumption, we get

$$z^2 = n^x + (9p)^y \equiv 1^x + 9^y \equiv 1^x + 1^y \equiv 2 \pmod 4.$$

This is impossible because $z^2 \equiv 0, 1 \pmod 4$. Hence the Diophantine equation $n^x + (9p)^y = z^2$ has no solution. □

Theorem 2.2. *Assume that* $n = 2$ *and* $p \equiv 3 \pmod 4$. *Then* (n, p, x, y, z) *is a solution for the Diophantine equation* $n^x + (9p)^y = z^2$ *if and only if* $(n, p, x, y, z) = (2, p, 3, 0, 3), (2, 7, 8, 2, 65)$ *or*

$$(n, p, x, y, z) \in \left\{(2, p, 0, 1, \sqrt{9p+1}) \mid \sqrt{9p+1} \in \mathbb{Z}\right\}.$$

Proof. Assume that (n, p, x, y, z) is a solution of the Diophantine equation $n^x + (9p)^y = z^2$. Thus $2^x + (9p)^y = z^2$.
Case 1: $x = 0$. Then we have $(9p)^y = z^2 - 1$. We consider the following four subcases:

If $y = 0$, then $z^2 = n^x + (9p)^y = 1 + 1 = 2$. Thus $z^2 = 2$ which is impossible.

If $y = 1$, then $z^2 = n^x + (9p)^y = 9p + 1$. We have $z = \sqrt{9p+1}$ where $\sqrt{9p+1}$ is integer.

If $y = 2$, then we have $(9p)^2 = z^2 - 1$. Then $9p = 0$ which is impossible.

If $y \geq 3$, then, by Mihăilescu's Theorem, this is impossible.
Summarizing, we have that $(n, p, x, y, z) = (2, p, 0, 1, \sqrt{9p+1})$ where $\sqrt{9p+1}$ is integer.
Case 2: $x = 1$. Then $z^2 = 2+(9p)^y \equiv 2+0 \equiv 2 \pmod 3$ which is impossible.
Case 3: $x \geq 2$. Then we have $4 \mid 2^x$, so that $2^x \equiv 0 \pmod 4$. Since 2^x is even and $(9p)^y$ is odd, $z^2 = 2^x + (9p)^y$ is odd. This implies that $z^2 \equiv 1 \pmod 4$. We get $z^2 = 2^x + (9p)^y \equiv (-1)^y \pmod 4$ because $p \equiv 3 \pmod 4$. If y is odd, then $z^2 \equiv -1 \equiv 3 \pmod 4$ which is impossible. Then y must be even.
Case 3.1: $y = 0$. Then we have $z^2 - 1 = 2^x$. Since $x \geq 2$, $(n, p, x, y, z) = (2, p, 3, 0, 3)$.
Case 3.2: $y \geq 2$. Then $y = 2k$ for some positive integer k. It follows that $z^2 - (9p)^{2k} = 2^x$ and so $(z - (9p)^k)(z + (9p)^k) = 2^x$. Then we have a non-negative integer w that satisfies $z - (9p)^k = 2^w$ and $z + (9p)^k = 2^{x-w}$. We have $2^{x-w} = z + (9p)^k > z - (9p)^k = 2^w$. Then $x - w > w$ and so $x - 2w > 0$. Moreover, we get $2(9p)^k = 2^{x-w} - 2^w, 2(9p)^k = 2^w(2^{x-2w} - 1)$ and $(9p)^k = 2^{w-1}(2^{x-2w} - 1)$. Since $(9p)^k$ is odd, we have 2^{w-1} must be odd. Thus $2^{w-1} = 1$. This implies that $w = 1$. Therefore, $2^{x-2} - (9p)^k = 1$. We consider the following two subcases:
Case 3.2.1: $k > 1$. We consider the following subcases:

If $x = 2$, then $(9p)^k = 0$. This is impossible.

If $x = 3$, then $(9p)^k = 1$. This implies that $k = 0$ which is a contradiction.

If $x > 3$, then, by Mihăilescu's Theorem, this is impossible.

Case 3.2.2: $k = 1$. Then we have $2^{x-2} - 1 = 9p$. We consider this equation modulo 9. We have $2^{x-2} \equiv 1 \pmod 9$.

If $x - 2 \equiv 1 \pmod 6$, then $2^{x-2} - 1 \equiv 2 - 1 \equiv 1 \pmod 9$, which is a contradiction.

If $x - 2 \equiv 2 \pmod 6$, then $2^{x-2} - 1 \equiv 4 - 1 \equiv 3 \pmod 9$ which is a contradiction.

If $x - 2 \equiv 3 \pmod 6$, then $2^{x-2} - 1 \equiv 8 - 1 \equiv 7 \pmod 9$ which is a contradiction.

If $x - 2 \equiv 4 \pmod 6$, then $2^{x-2} - 1 \equiv 16 - 1 \equiv 6 \pmod 9$ which is a contradiction.

If $x - 2 \equiv 5 \pmod 6$, then $2^{x-2} - 1 \equiv 32 - 1 \equiv 4 \pmod 9$ which is a contradiction.

If $x-2 \equiv 0 \pmod 6$, then $2^{x-2}-1 \equiv 64-1 \equiv 0 \pmod 9$. Thus, $x-2 \equiv 0 \pmod 6$. Let $x-2 = 6l$ for some positive integer l. It follows that $2^{6l}-1 = 9p$. Thus $(2^{2l} - 1)(2^{4l} + 2^{2l} + 1) = 9p$. We consider the following subcases:

Subcase $2^{2l} - 1 = 1$ and $2^{4l} + 2^{2l} + 1 = 9p$. We have that $2l = 1$ which is impossible.

Subcase $2^{2l} - 1 = 3$ and $2^{4l} + 2^{2l} + 1 = 3p$. We have that $l = 1$. Then $3p = 16+4+1 = 21$. Then $p = 7$. In this case, $(n, p, x, y, z) = (2, 7, 8, 2, 65)$.

Subcase $2^{2l} - 1 = 9$ and $2^{4l} + 2^{2l} + 1 = p$ is impossible.

Subcase $2^{2l} - 1 = p$ and $2^{4l} + 2^{2l} + 1 = 9$ is impossible.

Subcase $2^{2l} - 1 = 3p$ and $2^{4l} + 2^{2l} + 1 = 3$. Since $2^{4l} + 2^{2l} + 1 = 3$, $l = 0$. Then $3p = 0$ which is impossible.

Subcase $2^{2l} - 1 = 9p$ and $2^{4l} + 2^{2l} + 1 = 1$ is impossible. □

Remark. In Case 1 of the proof of Theorem 2.2, if we choose $p = 7$ or 11, then we have that $\sqrt{9p+1}$ is an integer. Therefore, $(n, p, x, y, z) = (2, 7, 0, 1, 8)$ and $(2, 11, 0, 1, 10)$ are solutions for the Diophantine equation $n^x + (9p)^y = z^2$.

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