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On 3-Distance Independent Sets of Graphs

Piyatida Boonsanong[1], **Pattarawan Singavananda**[1], **Ronnason Chinram**[2]

[1]Mathematics Program
Faculty of Science and Technology
Songkhla Rajabhat University
Songkhla 90000, Thailand

[2]Division of Computational Science
Faculty of Science
Prince of Songkla University
Hat Yai, Songkhla 90110, Thailand

email: piyatida.bo@skru.ac.th, pattarawan.pe@skru.ac.th,
ronnason.c@psu.ac.th

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Abstract

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A set $S \subseteq V(G)$ is a 3-distance independent set of G if $d_G(v,w) \neq 3$ for any two distinct vertices $v, w \in S$. The maximum cardinality of a 3-distance independent set of G, denoted by $\alpha^3(G)$, is called the 3-distance independence number of G. In this paper, we establish basic bounds for $\alpha^3(G)$, we characterize the graphs for which $\alpha^3(G)$ attains its extreme values, and we introduce the outer 3-distance independent vertex cover of a graph, obtaining a lower bound for its cardinality in terms of $\alpha^3(G)$.

Key words and phrases: 3-distance independent set, 3-distance independence number, 3-distance domination number, outer 3-distance independent vertex cover, trivial graph.
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The corresponding author is Ronnason Chinram.
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1 Introduction

Independence and domination are two of the most fundamental concepts in graph theory. A set $S \subseteq V(G)$ is independent if no two of its vertices are adjacent, and the maximum such cardinality is the independence number $\alpha(G)$; a set $D \subseteq V(G)$ dominates G if every vertex outside D is adjacent to some vertex of D, and the minimum such cardinality is the domination number $\gamma(G)$. See Haynes, Hedetniemi and Slater [1] for a comprehensive treatment of domination theory. Replacing adjacency, that is, distance one, by an arbitrary fixed distance leads to several natural variants. Meir and Moon [2] initiated the study of distance-based packing and covering parameters in trees. Natarajan and Ayyaswamy [3] introduced the hop domination number of a graph, where a vertex u hop-dominates a vertex v if $d_G(u, v) = 2$, and Abiad, Coutinho and Fiol [4] studied the k-independence number $\alpha_k(G)$, the maximum cardinality of a set of vertices that are pairwise at distance greater than k. Motivated by this framework, Hassan, Canoy and Aradais [5] introduced the hop independent set of a graph, requiring $d_G(u, v) \neq 2$ for distinct $u, v \in S$; this is one of the main motivations of the present paper. Hassan, Gamorez and Ahmad [6] introduced the outer-hop independent vertex cover of a graph, while Hassan, Iyah, Paradji, Gamorez and Ahmad [7] studied the 3-distance dominating set of a graph: a set $Q \subseteq V(G)$ is a 3-distance dominating set of G if every $u \in V(G) \setminus Q$ has some $v \in Q$ with $d_G(u, v) = 3$, and the minimum such cardinality is the 3-distance domination number $\gamma^3(G)$. A vertex v is a 3-distance neighbor of u in G if $d_G(u, v) = 3$; write $N^3_G(u) = \{v \in V(G) \mid d_G(v, u) = 3\}$ for the 3-distance open neighborhood of u. For $S \subseteq V(G)$, let $\langle S \rangle$ denote the subgraph of G induced by S. A graph with exactly one vertex is called trivial.

In this paper, we introduce the 3-distance independent set of a graph, that is, a set $S \subseteq V(G)$ in which no two distinct vertices are at distance exactly three, and we define $\alpha^3(G)$, the 3-distance independence number of G, as the maximum cardinality of such a set. We relate $\alpha^3(G)$ to $\gamma^3(G)$, establish bounds for $\alpha^3(G)$ and characterize the graphs attaining its extreme values, and extend the outer-hop independent vertex cover of [6] to the 3-distance setting.

2 Main results

Let G be an undirected graph with vertex set $V(G)$ and edge set $E(G)$. A set $S \subseteq V(G)$ is a 3-distance independent set of G if $d_G(v, w) \neq 3$ for any two distinct vertices $v, w \in S$. The maximum cardinality of a 3-distance independent set of G, denoted by $\alpha^3(G)$, is called the 3-distance independence number of G.

Proposition 2.1. *Let G be any graph. If S is a maximum 3-distance independent set of G, then S is a 3-distance dominating set. In particular, $\gamma^3(G) \leq \alpha^3(G)$.*

Proof. Let S be a maximum 3-distance independent set of G and let $v \in V(G) \setminus S$. If $d_G(v, w) \neq 3$ for all $w \in S$, then $S \cup \{v\}$ is a 3-distance independent set of G, contradicting the maximality of S. Thus, there exists $z \in S$ such that $d_G(v, z) = 3$. This shows that S is a 3-distance dominating set of G. Therefore $\gamma^3(G) \leq \alpha^3(G)$. $\square$

Theorem 2.2. *Let G be a graph and let S be a 3-distance independent set of G. If x and y lie in the same component of $\langle S \rangle$, then $d_G(x, y) \leq 2$.*

Proof. Let $x, y \in S$ lie in the same component of $\langle S \rangle$. Let $v_0, v_1, \ldots, v_m$ be a path in $\langle S \rangle$ with $v_0 = x$ and $v_m = y$; note that each $v_i \in S$. Since v_i and v_{i+1} are adjacent in G for each i, the quantity $d_G(x, v_i)$ changes by at most one as i increases by one, and $d_G(x, v_0) = 0$. If $d_G(x, y) = d_G(x, v_m) \geq 3$, then by this discrete intermediate value property there is some index i with $d_G(x, v_i) = 3$. Since $x, v_i \in S$, this contradicts the assumption that S is a 3-distance independent set of G. Hence, $d_G(x, y) \leq 2$. $\square$

Theorem 2.3. *Let G be a nonempty graph. Then $1 \leq \alpha^3(G) \leq |V(G)|$. Moreover, each of the following holds:*

(1) $\alpha^3(G) = 1$ if and only if G is trivial.

(2) $\alpha^3(G) = |V(G)|$ if and only if every component C of G satisfies $diam(C) \leq 2$.

Proof. Since every set consisting of a single vertex is a 3-distance independent set of G, we have $1 \leq \alpha^3(G) \leq |V(G)|$.

(1) Suppose that $\alpha^3(G) = 1$ and, to the contrary, that G is not trivial, so $|V(G)| \geq 2$. Let $u \in V(G)$ be such that $\{u\}$ is a maximum 3-distance independent set of G, and let $v \in V(G) \setminus \{u\}$. If u and v lie in the same component of G, let w be a neighbor of u; then $d_G(u, w) = 1 \neq 3$, so $\{u, w\}$ is a 3-distance independent set of G of cardinality two, contradicting $\alpha^3(G) = 1$. If u and v lie in different components of G, then $d_G(u, v) \neq 3$, so $\{u, v\}$ is a 3-distance independent set of G of cardinality two, again a contradiction. Hence, G is trivial. The converse is clear.

(2) Suppose that $\alpha^3(G) = |V(G)|$. Then $V(G)$ is a 3-distance independent set of G, so $d_G(u, v) \neq 3$ for all distinct $u, v \in V(G)$. In particular, no two vertices lying in the same component of G are at distance 3. Let C be any component of G and suppose, to the contrary, that $diam(C) \geq 3$. Choose $x, y \in C$ with $d_G(x, y) = diam(C)$ and let $x = u_0, u_1, \ldots, u_k = y$ (where $k = diam(C) \geq 3$) be a shortest x-y path in G. Since this path is geodesic, $d_G(x, u_i) = i$ for each i, so in particular $d_G(x, u_3) = 3$, contradicting the fact that no two vertices of G are at distance 3. Hence $diam(C) \leq 2$ for every component C of G. Conversely, if every component C of G satisfies $diam(C) \leq 2$, then no two vertices of G are at distance exactly 3, so $V(G)$ is a 3-distance independent set of G and $\alpha^3(G) = |V(G)|$. □

For any graph G, let $\delta^3(G) = \min\{|N_G^3(v)| \mid v \in V(G)\}$.

Theorem 2.4. *For any graph G, $\alpha^3(G) \leq |V(G)| - \delta^3(G)$.*

Proof. Let S be a maximum 3-distance independent set of G and let $v \in S$. By definition, $\delta^3(G) \leq |N_G^3(v)|$. Since S is a 3-distance independent set of G and $v \in S$, no vertex of $S \setminus \{v\}$ lies in $N_G^3(v)$, so $N_G^3(v) \subseteq V(G) \setminus S$. Hence, $\delta^3(G) \leq |N_G^3(v)| \leq |V(G)| - |S| = |V(G)| - \alpha^3(G)$. Therefore, $\alpha^3(G) \leq |V(G)| - \delta^3(G)$. □

A non-empty set $P \subseteq V(G)$ is called a vertex cover of G if every edge of G is incident to some vertex of P. The vertex cover number of G, denoted by $\beta(G)$, is the minimum cardinality of a vertex cover of G. Extending the notion of an outer-hop independent vertex cover introduced by Hassan, Gamorez and Ahmad [6], a non-empty set $P \subseteq V(G)$ is called an outer 3-distance independent vertex cover of G if P is a vertex cover of G and $V(G) \setminus P$ is empty or a 3-distance independent set of G. The outer 3-distance independent vertex cover number of G, denoted by $\beta^3_{outer}(G)$, is the minimum cardinality among all outer 3-distance independent vertex covers of G.

Remark 2.5. *Let G be a graph and let $Q \subseteq V(G)$ be an outer 3-distance independent vertex cover of G. Then*

(i) Q is not necessarily a 3-distance independent set of G;

(ii) $1 \leq \beta(G) \leq \beta^3_{outer}(G) \leq |Q| \leq |V(G)|$.

Theorem 2.6. *Let G be a graph. Then $\beta^3_{outer}(G) \geq |V(G)| - \alpha^3(G)$.*

Proof. Let G be a graph and let P be a minimum outer 3-distance independent vertex cover of G. Then $V(G) \setminus P$ is a 3-distance independent set of G. Thus $\alpha^3(G) \geq |V(G) \setminus P| \geq |V(G)| - |P|$. It follows that $|P| \geq |V(G)| - \alpha^3(G)$. Since P is the minimum outer 3-distance independent vertex cover of G, we have $\beta^3_{outer}(G) = |P| \geq |V(G)| - \alpha^3(G)$. □

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