# Left and right magnifying elements in some generalized partial transformation semigroups

Ronnason Chinram, Samruam Baupradist, Naveed Yaqoob & Pattarawan Petchkaew

COMMUNICATIONS IN ALGEBRA®
2021, VOL. 49, NO. 7, 3176–3191
https://doi.org/10.1080/00927872.2021.1890104



# Left and right magnifying elements in some generalized partial transformation semigroups

Ronnason Chinram[a], Samruam Baupradist[b], Naveed Yaqoob[c], and Pattarawan Petchkaew[d]

[a]Algebra and Applications Research Unit, Division of Computational Science, Faculty of Science, Prince of Songkla University, Hat Yai, Songkhla, Thailand; [b]Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok, Thailand; [c]Department of Mathematics and Statistics, Riphah International University, Islamabad, Pakistan; [d]Mathematics Program, Faculty of Science and Technology, Songkhla Rajabhat University, Songkhla, Thailand

**ABSTRACT**

An element $a$ of a semigroup $S$ is called left [right] magnifying if there exists a proper subset $M$ of $S$ such that $S = aM$ [$S = Ma$]. Let $P(X)$ be a semigroup of all partial transformations on a set $X$ under the composition of maps. A number of results concerning the necessary and sufficient conditions for elements in some interesting generalized semigroups of partial transformations to be left or right magnifiers are presented.

**ARTICLE HISTORY**

Received 3 May 2018
Revised 5 January 2021
Communicated by J. Zhang

**KEYWORDS**

Left [right] magnifying elements; partial transformations preserving partitions; partial transformation semigroups; partial transformations with invariant set; partial transformations with restricted range

**2020 MATHEMATICS SUBJECT CLASSIFICATION**

20M10; 20M20

## 1. Introduction and preliminary

The notions of left and right magnifying elements of semigroups were introduced by Ljapin in [10]. An element $a$ of a semigroup $S$ is called left [right] magnifying if there exists a proper subset $M$ of $S$ such that $S = aM$ $[S = Ma]$. Some research of magnifying elements in semigroups can be seen in [1–9, 11–13]. In [1], Catino and Migliorini gave necessary and sufficient conditions for any semigroups to contain left and right magnifying elements. In [11], Magill, Jr. investigated necessary and sufficient conditions for elements in the transformation semigroup $T(X)$ to be left or right magnifying and applied those conditions for elements in linear transformation semigroups and in semigroups of all continuous selfmaps to be left or right magnifying. Next, Gutan verified about the semigroups with strong and non-strong magnifying elements in [7] and after that he found that every semigroups containing magnifying elements is factorizable in [5]. Furthermore, Gutan also studied about the semigroups with magnifiers admitting minimal subsemigroup in [6] and after that he and Kisielewicz investigated about semigroups with good and

CONTACT Ronnason Chinram ronnason.c@psu.ac.th Algebra and Applications Research Unit, Division of Computational Science, Faculty of Science, Prince of Songkla University, Hat Yai, Songkhla, Thailand.

bad magnifying in [8]. It is well-known that $T(X) = \{f : X \to X | f \text{ is a function}\}$ is a semigroup under the composition of functions and it is called the full transformation semigroup on a set $X$. Transformation semigroups play an important role in semigroup theory since it is well-known that every semigroup is isomorphic to a subsemigroup of a suitable full transformation semigroup. Recently, necessary and sufficient conditions for elements in some generalized full transformation semigroups to be left or right magnifying were given in [2–4]. These are our motivation to do this research.

Let $P(X)$ be the partial transformation semigroup on a set $X$, that is, $P(X)$ is the set of all functions from a subset of $X$ to a set $X$ under the composition of functions. Throughout this paper, we write functions from the right, $(x)f$ rather than $f(x)$, and compose from the left to the right, $(x)(fg)$ rather than $(g \circ f)(x)$ for the elements $f$, $g$ in the partial transformation semigroup $P(X)$ and $x \in X$. Let $Y$ be a fixed subset of a set $X$. Let

$$P(X, Y) = \{f \in P(X) | ran\ f \subseteq Y\},$$
$$V(X, Y) = \{f \in P(X) | (Y)f \subseteq Y\}$$

and

$$S(X, Y) = \{f \in P(X) | (Y)f \subseteq Y \text{ and } (X \setminus Y)f \subseteq X \setminus Y\}.$$

Note that: $P(X, Y), V(X, Y)$ and $S(X, Y)$ are subsemigroups of $P(X)$. If $Y = X$, we have that $P(X, Y) = V(X, Y) = S(X, Y) = P(X)$. Then $P(X, Y), V(X, Y)$ and $S(X, Y)$ are generalizations of the partial transformation semigroups. $P(X, Y), V(X, Y)$ and $S(X, Y)$ are called the semigroup of partial transformations with restricted range, the semigroup of partial transformations with invariant set and the semigroup of partial transformations preserving partitions, respectively.

Our aim in this paper is to give the necessary and sufficient conditions for elements in semigroups $P(X, Y), V(X, Y)$ and $S(X, Y)$ to be left or right magnifying.

## 2. Magnifying elements in semigroups of partial transformations with restricted range

In this section, we investigate about the necessary and sufficient conditions for elements in $P(X, Y)$ to be left and right magnifying.

### 2.1. Right magnifying elements

**Lemma 1.** *If $f$ is a right magnifying element in $P(X, Y)$, then $f$ is onto a set $Y$.*

*Proof.* Assume that $f$ is a right magnifying element in $P(X, Y)$. Then there exists a proper subset $M$ of $P(X, Y)$ such that $Mf = P(X, Y)$. Since $id_Y \in P(X, Y)$, there exists a function $h \in M$ such that $hf = id_Y$. This implies that $f$ is onto a set $Y$. □

**Lemma 2.** *Let $f \in P(X, Y)$ be onto a set $Y$.*

1. *If $(y)f^{-1} \cap Y = \emptyset$ for some $y \in Y$, then $f$ is not right magnifying.*
2. *If $Y \not\subseteq dom\ f$ and $(y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$, then $f$ is right magnifying.*
3. *If $Y \subseteq dom\ f$ and $|(y)f^{-1} \cap Y| = 1$ for all $y \in Y$, then $f$ is not right magnifying.*
4. *If $Y \subseteq dom\ f, (y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ and $|(y)f^{-1} \cap Y| > 1$ for some $y \in Y$, then $f$ is right magnifying.*

*Proof.* 1. Assume that $(y)f^{-1} \cap Y = \emptyset$ for some $y \in Y$. Let $y_0 \in Y$ be such that $(y_0)f^{-1} \cap Y = \emptyset$. Let $g \in P(X, Y)$ be such that $dom\ g = X$ and $(x)g = y_0$ for all $x \in X$. Then for all $h \in P(X, Y)$,

$y_0 \notin ran\ hf$. This implies that there is no $h \in P(X, Y)$ such that $hf = g$. Therefore, $f$ is not right magnifying.

2. Assume that $Y \not\subseteq dom\ f$. Let $y_0 \in Y$ be such that $y_0 \notin domf$. Let $M = \{h \in P(X, Y) | y_0 \notin ran\ h\}$. Then $M \neq P(X, Y)$. Let $g$ be any function in $P(X, Y)$. Since $f$ is onto a set $Y$ and $(y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$, there exists for each $x \in dom\ g, y_x \in Y$ such that $(y_x)f = (x)g$. Define a function $h \in P(X, Y)$ by $(x)h = y_x$ for all $x \in dom\ g$. Thus $dom\ h = dom\ g$. Since $y_0 \notin dom\ f, y_0 \notin ran\ h$. Hence $h \in M$. For all $x \in dom\ g$, we obtain

$$(x)hf = ((x)h)f = (y_x)f = (x)g.$$

Then $hf = g$, this implies that $Mf = P(X, Y)$. Therefore, $f$ is right magnifying.

3. Assume that $Y \subseteq dom\ f$ and $|(y)f^{-1} \cap Y| = 1$ for all $y \in Y$. Then $f|_Y$ is bijective. Suppose $f$ is right magnifying. Then there exists a proper subset $M$ of $P(X, Y)$ such that $Mf = P(X, Y)$. Hence, $Mf = P(X, Y)f$. Since $f|_Y$ is bijective, $M = P(X, Y)$, a contradiction. Then $f$ is not right magnifying.

4. Assume that $Y \subseteq dom\ f, (y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ and $|(y)f^{-1} \cap Y| > 1$ for some $y \in Y$. Let $M = \{h \in P(X, Y) | h$ is not onto a set $Y\}$. Then $M \neq P(X, Y)$. Let $g$ be any function in $P(X, Y)$. Since $f$ is onto a set $Y$ and $(y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$, there exists for each $x \in X$, an element $y_x \in Y$ such that $(y_x)f = (x)g$ (if $(x_1)g = (x_2)g$, we must choose $y_{x_1} = y_{x_2}$). Define a function $h \in P(X, Y)$ by $(x)h = y_x$ for all $x \in dom\ g$. We claim that $h$ is not onto a set $Y$. Since $|(y)f^{-1} \cap Y| > 1$ for some $y \in Y$, there exist an element $y' \in Y$ and distinct elements $y_1, y_2 \in Y$ such that $(y_1)f = (y_2)f = y'$. If $y' \notin ran\ g$, we have $y_1, y_2 \notin ran\ h$. If $y' \in ran\ g$, there is only one between $y_1$ and $y_2$ in $ran\ h$. Then $h$ is not onto a set $Y$. Hence $h \in M$ and for all $x \in dom\ g = dom\ h$, we gain

$$(x)hf = (y_x)f = (x)g.$$

Then $hf = g$, hence $Mf = P(X, Y)$. Therefore, $f$ is right magnifying. □

**Example 1.** Consider $X = \mathbb{N}$ and $Y = 2\mathbb{N}$.

1. Let $f \in P(X, Y)$ by $(1)f = (3)f = 2$ and $(2x)f = 2x$ for all positive integer $x > 1$, that is,

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & - & 2 & 4 & - & 6 & - & 8 & \cdots \end{pmatrix}.$$

Then $f$ is onto a set $Y$ and is such that $(2)f^{-1} \cap Y = \emptyset$. By Lemma 2(1), $f$ is not right magnifying.

2. Let $f \in P(X, Y)$ by $(1)f = (3)f = 2$ and $(2x)f = 2x - 2$ for all positive integer $x > 1$, that is,

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & - & 2 & 2 & - & 4 & - & 6 & \cdots \end{pmatrix}.$$

Then $f$ is onto a set $Y$ and $Y \not\subseteq dom\ f$ since $2 \notin dom\ f$ and $(y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$. Let $M = \{h \in P(X, Y) | 2 \notin ran\ h\}$. Let $g \in P(X, Y)$ be any function. By Lemma 2(2), there exists $h \in M$ such that $hf = g$. For example, if $g \in P(X, Y)$ such that $(x)g = 2x$ for all $x \in X$, that is,

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & \cdots \end{pmatrix}.$$

Define a function $h \in P(X, Y)$ by $(x)h = 2x + 2$ for all $x \in X$, that is,

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & \cdots \end{pmatrix}.$$

So $h \in M$ and we acquire

$$hf = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & \cdots \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & - & 2 & 2 & - & 4 & - & 6 & \cdots \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & \cdots \end{pmatrix} = g.$$

3. Let $f \in P(X, Y)$ by $(1)f = (2)f = 2$ and $(2x)f = 2x$ for all positive integer $x > 1$, that is,
$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 2 & - & 4 & - & 6 & - & 8 & \cdots \end{pmatrix}.$$
Then $f$ is onto a set $Y$, $Y \subseteq dom\ f$ and $|(y)f^{-1} \cap Y| = 1$ for all $y \in Y$. By Lemma 2(3), $f$ is not right magnifying.
4. Let $f \in P(X, Y)$ by $(1)f = (3)f = 4$ and $(2)f = (4)f = 2$ and $(2x)f = 2x - 2$ for all positive integer $x > 2$, that is,
$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 4 & 2 & 4 & 2 & - & 4 & - & 6 & \cdots \end{pmatrix}.$$

Then $f$ is onto a set $Y$, $(y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ and $|(y)f^{-1} \cap Y| > 1$ for some $y \in Y$. Let $M = \{h \in P(X, Y) | h \text{ is not onto a set } Y\}$. Let $g \in P(X, Y)$ be any function. By Lemma 2(4), there exists $h \in M$ such that $hf = g$. For instance, if $g \in P(X, Y)$ such that $(x)g = 2x$ for all $x \in X$, that is,

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & \cdots \end{pmatrix}.$$

Define a function $h \in P(X, Y)$ by $(1)h = 2$ and $(x)h = 2x + 2$ for all positive integer $x > 1$, that is,

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & \cdots \end{pmatrix}.$$

Hence $h \in M$, and so

$$hf = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & \cdots \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 4 & 2 & 4 & 2 & - & 4 & - & 6 & \cdots \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & \cdots \end{pmatrix} = g.$$

**Theorem 3.** *A function $f$ is right magnifying in P(X, Y) if and only if $f$ is onto a set $Y$ such that either*

1. *$Y \not\subseteq dom\ f$ and $(y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ or*
2. *$Y \subseteq dom\ f, (y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ and $|(y)f^{-1} \cap Y| > 1$ for some $y \in Y$.*

*Proof.* This follows by Lemmas 1 and 2. □

**Corollary 4.** *A function $f$ is right magnifying in P(X) if and only if $f$ is onto a set $Y$ and is such that either*

1. *dom* $f \neq X$ *or*
2. *dom* $f = X$ *and* $f$ *is not one-to-one.*

*Proof.* This follows by Theorem 3. □

Although we know that the definition of left and right magnifying have the similar structure, but the conditions for elements in $P(X, Y)$ to be left or right magnifying are different as we show in the part of right magnifying as above and the part of left magnifying as follows.

### 2.2. Left magnifying elements

**Lemma 5.** *If* $f$ *is a left magnifying element of* $P(X, Y)$ *and* $Y \neq \emptyset$, *then dom* $f = X$.

*Proof.* Let $f$ be a left magnifying element in $P(X, Y)$. So there exists a proper subset $M$ of $P(X, Y)$ such that $fM = P(X, Y)$. Let $y_0 \in Y$ and $g \in P(X, Y)$ be such that $dom\ g = X$ and $(x)g = y_0$ for all $x \in X$. Since $g \in P(X, Y) = fM, dom\ f = X$. □

**Lemma 6.** *If* $|Y| < |X|$, *then* $P(X, Y)$ *has no left magnifying element.*

*Proof.* If $Y = \emptyset$, then $|P(X, Y)| = 1$, it is obvious that $P(X, Y)$ has no left magnifying element. Assume $Y \neq \emptyset$. Let $f$ be a left magnifying element in $P(X, Y)$. By Lemma 5, $dom\ f = X$. Since $|Y| < |X|$, $f$ is not one-to-one. So there exist $y \in Y$ and $x_1, x_2 \in X$ such that $(x_1)f = (x_2)f = y$. Define $g \in P(X, Y)$ by $dom\ g = \{x_1\}$ and $(x_1)g = y$. Clearly, there is no $h \in P(X, Y)$ such that $fh = g$. This implies that there is no a proper subset $M$ of $P(X, Y)$ such that $fM = P(X, Y)$. Hence, $P(X, Y)$ has no left magnifying element. □

**Lemma 7.** *Assume* $|Y| = |X|$. *If* $f$ *is a left magnifying element in* $P(X, Y)$, *then* $f$ *is one-to-one.*

*Proof.* Assume that $f$ is a left magnifying element in $P(X, Y)$. Then there exists a proper subset $M$ of $P(X, Y)$ such that $fM = P(X, Y)$. Since $|X| = |Y|$, there exists a one-to-one function $h$ from $X$ onto $Y$ in $P(X, Y)$. Therefore, there exists $g \in M$ such that $fg = h$. This implies that $f$ is one-to-one. □

**Lemma 8.** *Assume that* $|Y| = |X|$ *but* $Y \neq X$. *If* $f \in P(X, Y)$ *is one-to-one and dom* $f = X$, *then* $f$ *is a left magnifying element in* $P(X, Y)$.

*Proof.* Assume that $f \in P(X, Y)$ is one-to-one and $dom\ f = X$. Let $y_0 \in Y$ and $M = \{h \in P(X, Y) | (x)h = y_0$ for all $x \notin ran\ f\}$. Claim that $fM = P(X, Y)$. Let $g \in P(X, Y)$. Define $h \in P(X, Y)$ by for all $x \in X$,

$$(x)h = \begin{cases} (x')g & \text{if } x \in ran\ f \text{ and } (x')f = x, \\ y_0 & \text{if } x \notin ran\ f. \end{cases}$$

Then $h \in M$ and for $x \in X$, and so

$$(x)fh = ((x)f)h = (x)g.$$

Then $fh = g$, this implies that $fM = P(X, Y)$. Hence $f$ is left magnifying in $P(X, Y)$. □

**Example 2.** Consider $X = \mathbb{N}$ and $Y = 2\mathbb{N}$. Let $f \in P(X, Y)$ by $(x)f = 2x$ for all positive integer $x$, that is,

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & \cdots \end{pmatrix}.$$

Then $dom\ f = X$ and $f$ is one-to-one. Let $M = \{h \in P(X, Y) | (2x+1)h = 2 \text{ for all } x \in \mathbb{N}\}$. Let $g \in P(X, Y)$ be any function. Define a function $h \in P(X, Y)$ by $(2x)h = (x)g$ and $(2x-1)h = 2$ for all positive integer $x$. Thus $h \in M$ and for $x \in X$, and then

$$(x)fh = ((x)f)h = (2x)h = (x)g.$$

Hence $fh = g$. For example, if $g \in P(X, Y)$ such that $(x)g = 4x$ for all $x \in X$, that is,

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & \cdots \end{pmatrix}.$$

Define a function $h \in P(X, Y)$ by $(2x)h = 4x$ and $(2x-1)h = 2$ for all $x \in X$, that is,

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 4 & 2 & 8 & 2 & 12 & 2 & 16 & \cdots \end{pmatrix}.$$

So $h \in M$ and we acquire

$$\begin{aligned} fh &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & \cdots \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 4 & 2 & 8 & 2 & 12 & 2 & 16 & \cdots \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & \cdots \end{pmatrix} = g. \end{aligned}$$

**Theorem 9.** *Assume $|X| = |Y|$ and $Y \neq X$. A function f is left magnifying of P(X, Y) if and only if f is one-to-one and dom $f = X$.*

*Proof.* This follows from Lemmas 7 and 8. □

**Theorem 10.** *A function f in P(X) is a left magnifying element if and only if dom $f = X$ and f is one-to-one but not onto a set X.*

*Proof.* Assume that $f$ is one-to-one but not onto a set $X$ and $dom\ f = X$. Let $y_0 \in X$ and $M = \{h \in P(X) | (x)h = y_0 \text{ for all } x \notin ran\ f\}$. We claim that $fM = P(X)$. For, let $g \in P(X)$. Define a function $h \in P(X)$ by for all $x \in X$,

$$(x)h = \begin{cases} (x')g & \text{if } x \in ran\ f \text{ and } (x')f = x, \\ y_0 & \text{if } x \notin ran\ f. \end{cases}$$

Then $h \in M$ and for $x \in X$, we have

$$(x)fh = ((x)f)h = (x)g.$$

Hence $fh = g$, and so $fM = P(X)$. Since $M$ is a proper subset of $P(X)$, $f$ is a left magnifying element in $P(X)$. Conversely, assume that $f$ is a left magnifying element in $P(X)$. By Lemma 5, $dom\ f = X$. By Lemma 7, $f$ is one-to-one. Assume that $f$ is onto a set $X$. Since $f$ is bijective, its inverse function $f^{-1}$ exists. Since $f$ is a left magnifying element in $P(X)$, there exists a proper subset $M$ of $P(X)$ such that $fM = P(X)$. This implies that $fM = fP(X)$ and $P(X) = f^{-1}fP(X) = f^{-1}fM = M$, a contradiction, and hence $f$ is not onto a set $X$. □

## 3. Magnifying elements of semigroups of partial transformations with invariant set

In this section, we study about the necessary and sufficient conditions for elements in $V(X, Y)$ to be left and right magnifying.

### 3.1. Right magnifying elements

**Lemma 11.** *If $f$ is a right magnifying element in $V(X, Y)$, then $f$ is onto a set $X$.*

*Proof.* The proof is similar to Lemma 1 by the fact that $id_X \in V(X, Y)$. □

**Lemma 12.** *Let $f \in V(X, Y)$ be onto a set $X$.*

1. *If $(y)f^{-1} \cap Y = \emptyset$ for some $y \in Y$, then $f$ is not right magnifying.*
2. *If $Y \nsubseteq dom\ f$ and $(y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$, then $f$ is right magnifying.*
3. *If $Y \subseteq dom\ f, (y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ and $f$ is one-to-one, then $f$ is not right magnifying.*
4. *If $Y \subseteq dom\ f, (y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ and $f$ is not one-to-one, then $f$ is right magnifying.*

*Proof.* 1. Assume that $(y)f^{-1} \cap Y = \emptyset$ for some $y \in Y$. Let $y' \in Y$ be such that $(y')f^{-1} \cap Y = \emptyset$. Define a function $g \in V(X, Y)$ be such that $dom\ g = X$ and $(x)g = y'$ for all $x \in X$. Thus, there is no $h \in V(X, Y)$ such that $hf = g$. Hence $f$ is not right magnifying.

2. Assume that $Y \nsubseteq dom\ f$. Let $y_0 \in Y$ be such that $y_0 \notin domf$. Let $M = \{h \in V(X, Y) | y_0 \notin ran\ h\}$. Then $M \neq V(X, Y)$. Let $g$ be any function in $V(X, Y)$. Since $f$ is onto a set $X$ and $(y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$, we can choose for each $x \in dom\ g$ such that $(x)g \in Y, y_x \in Y$ such that $(y_x)f = (x)g$ and for each $x \in dom\ g$ such that $(x)g \in X \setminus Y$, we can choose $y_x \in X$ such that $(y_x)f = (x)g$. Define a function $h \in V(X, Y)$ by $dom\ h = dom\ g$. and $(x)h = y_x$ for all $x \in dom\ g$. Note that if $x \in Y$, then $(x)g \in Y$, and so $y_x \in Y$. Since $y_0 \notin dom\ f, y_0 \notin ran\ h$. Hence $h \in M$. For all $x \in dom\ g$, we obtain that

$$(x)hf = ((x)h)f = (y_x)f = (x)g.$$

Then $hf = g$, and hence $Mf = V(X, Y)$. Therefore, $f$ is right magnifying.

3. Assume that $Y \subseteq dom\ f, (y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ and $f$ is one-to-one. Then $f$ and $f|_Y$ are bijective. Assume $f$ is right magnifying. Then there exists a proper subset $M$ of $V(X, Y)$ such that $Mf = V(X, Y)$. Hence $Mf = V(X, Y)f$. Since $f$ and $f|_Y$ are bijective, $M = V(X, Y)$, a contradiction. Then $f$ is not right magnifying.

4. Assume that $(y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$. Let $M = \{h \in V(X, Y) | h$ is not onto a set $X\}$. Then $M \neq V(X, Y)$. Let $g$ be any function in $V(X, Y)$. Since $f$ is onto a set $X$ and $(y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$, we can choose for each $x \in dom\ g$ such that $(x)g \in Y, y_x \in Y$ such that $(y_x)f = (x)g$ (if $(x_1)g = (x_2)g$, we must choose $y_{x_1} = y_{x_2}$) and for each $x \in dom\ g$ such that $(x)g \in X \setminus Y$, we can choose $y_x \in X$ such that $(y_x)f = (x)g$ (if $(x_1)g = (x_2)g$, we must choose $y_{x_1} = y_{x_2}$). Define a function $h \in V(X, Y)$ by $dom\ h = dom\ g$ and $(x)h = y_x$ for all $x \in dom\ g$. Note that if $x \in Y$, then $(x)g \in Y$, and so $y_x \in Y$. Next, we claim that $h$ is not onto a set $X$. Since $f$ is not one-to-one, there exist an element $x' \in X$ and distinct elements $x_1, x_2 \in X$ such that $(x_1)f = (x_2)f = x'$. If $x' \notin ran\ g$, we have $x_1, x_2 \notin ran\ h$. If $x' \in ran\ g$, there is only one between $x_1$ and $x_2$ in $ran\ h$. Then $h$ is not onto a set $X$. Hence $h \in M$ and for all $x \in X$, and so $(x)hf = ((x)h)f = (y_x)f = (x)g$. Thus $hf = g$, and hence $Mf = V(X, Y)$. Therefore, $f$ is right magnifying. □

**Example 3.** Consider $X = \mathbb{N}$ and $Y = 2\mathbb{N}$.

1. Let $f \in V(X, Y)$ by $(1)f = 1, (3)f = 2, (4)f = 4$ and $(x)f = x - 2$ for all positive integer $x > 4$, that is,

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & - & 2 & 4 & 3 & 4 & 5 & 6 & \cdots \end{pmatrix}.$$

   Then $f$ is onto a set $X$ such that $(2)f^{-1} \cap Y = \emptyset$. By Lemma 12(1), $f$ is not right magnifying.
2. Let $f \in V(X, Y)$ by $(1)f = 1, (3)f = 2$ and $(x)f = x - 2$ for all positive integer $x > 3$, that is,

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & - & 2 & 2 & 3 & 4 & 5 & 6 & \cdots \end{pmatrix}.$$

   Then $f$ is onto a set $X$, $Y \nsubseteq dom\ f$ and $(y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$. Let $M = \{h \in P(X, Y) | 2 \notin ran\ h\}$. Let $g \in V(X, Y)$ be any function. By Lemma 12(2), there exists $h \in M$ such that $hf = g$. For example, Let $g \in P(X, Y)$ be such that $(x)g = 2x$ for all $x \in X$, that is,

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & \cdots \end{pmatrix}.$$

   Define a function $h \in P(X, Y)$ by $(x)h = 2x + 2$ for all $x \in X$, that is,

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & \cdots \end{pmatrix}.$$

   So $h \in M$ and we have

$$hf = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & \cdots \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & - & 2 & 2 & 3 & 4 & 5 & 6 & \cdots \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & \cdots \end{pmatrix} = g.$$

3. We gain that $id_X$ is onto a set $X$, $Y \subseteq dom\ f, (y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ and $f$ is one-to-one. Then $id_X$ is not right magnifying by Lemma 12(3).
4. Let $f \in V(X, Y)$ by $(1)f = 1$ and $(2)f = 2$, and $(x)f = x - 2$ for all positive integers $x > 3$, that is,

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 1 & 2 & - & 2 & 3 & 4 & 5 & \cdots \end{pmatrix}.$$

Then $f$ is onto a set $X$ but not one-to-one and $(y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$. Let $M = \{h \in V(X, Y) | h$ is not onto a set $X\}$. In Lemma 12(4), for any $g \in V(X, Y)$, we have $h \in V(X, Y)$ such that $hf = g$. For example, if $g \in V(X, Y)$ such that $(x)g = 2x$ for all $x \in X$, that is,

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & \cdots \end{pmatrix}.$$

Define a function $h \in V(X, Y)$ by $(1)h = 2$ and $(x)h = 2x + 2$ for all $x > 1$, that is,

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 2 & 6 & 8 & 10 & 12 & 14 & 16 & \cdots \end{pmatrix}.$$

Then $h \in M$ and we gain

$$hf = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 2 & 6 & 8 & 10 & 12 & 14 & 16 & \cdots \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 1 & 2 & - & 2 & 3 & 4 & 5 & \cdots \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & \cdots \end{pmatrix} = g.$$

**Theorem 13.** *Let $f \in V(X, Y)$. Then $f$ is right magnifying in $V(X, Y)$ if and only if $f$ is onto a set $X$ such that either*

1. *$Y \nsubseteq dom\ f$ and $(y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ or*
2. *$Y \subseteq dom\ f, (y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ and $f$ is not one-to-one.*

*Proof.* This follows by Lemmas 11 and 12. □

**Corollary 14.** *Let $f \in P(X)$. Then $f$ is right magnifying in $P(X)$ if and only if $f$ is onto a set $X$ such that either*

1. *$dom\ f \neq X$ or*
2. *$dom\ f = X$ and $f$ is not one-to-one.*

*Proof.* This follows by Theorem 13. □

Currently, we see that the conditions for elements in $P(X, Y)$ and elements in $V(X, Y)$ to be right magnifying are parallel in some sense but they are not just the same. Moreover, as a result of Section 2, we know that the conditions for elements in $P(X, Y)$ to be left or right magnifying are different, it is the same for $V(X, Y)$.

### 3.2. Left magnifying elements

**Lemma 15.** *If $f$ is a left magnifying element of $V(X, Y)$ and $Y \neq \emptyset$, then $dom\ f = X$.*

*Proof.* The proof is similar to Lemma 5. □

**Lemma 16.** *If $f$ is a left magnifying element in $V(X, Y)$, then $f$ is one-to-one.*

*Proof.* Assume that $f$ is a left magnifying element in $V(X, Y)$. Then there exists a proper subset $M$ of $V(X, Y)$ such that $fM = V(X, Y)$. Since $id_X \in V(X, Y)$, there exists $g \in M$ such that $fg = id_X$. This implies that $f$ is one-to-one. □

**Lemma 17.** *Let $f \in V(X, Y)$. If $(y)f^{-1} \nsubseteq Y$ for some $y \in Y \cap ran\ f$, then $f$ is not a left magnifying element in $V(X, Y)$.*

*Proof.* Let $y' \in Y \cap ran\ f$ be such that $(y')f^{-1} \nsubseteq Y$. Then there exists $x' \in X \setminus Y$ such that $(x')f = y'$. We have $id_X \in V(X, Y)$ but there is no a function $h \in V(X, Y)$ such that $(x')fh = x'$. Hence $f$ is not left magnifying. □

**Lemma 18.** *Let $f \in V(X, Y)$ be such that $dom\ f = X$. If $f$ is bijective and $(y)f^{-1} \subseteq Y$ for all $y \in Y \cap ran\ f$, then $f$ is not a left magnifying element in $V(X, Y)$.*

*Proof.* Assume that $f$ is left magnifying. Then there exists a proper subset $M$ of $V(X, Y)$ such that $fM = V(X, Y)$. Then $fM = fV(X, Y)$. By assumption, we have $f|_Y$ is bijective. This implies that $f^{-1} \in V(X, Y)$, and so $M = f^{-1}fM = f^{-1}fV(X, Y) = V(X, Y)$, a contradiction. Hence $f$ is not a left magnifying element in $V(X, Y)$. □

**Lemma 19.** *Let $f \in V(X, Y)$ be such that $dom\ f = X$. If $f$ is one-to-one but not onto a set $X$ and $(y)f^{-1} \subseteq Y$ for all $y \in Y \cap ran\ f$, then $f$ is a left magnifying element in $V(X, Y)$.*

*Proof.* Assume that $f$ is one-to-one but not onto a set $X$ and $(y)f^{-1} \subseteq Y$ for all $y \in Y \cap ran\ f$. Let $y' \in Y$ and $M = \{h \in V(X, Y) | (x)h = y' \text{ for all } x \notin ran\ f\}$. Then $M \neq V(X, Y)$. We claim that $fM = V(X, Y)$. Let $g \in V(X, Y)$. Define a function $h \in V(X, Y)$ by for all $x \in X$,

$$(x)h = \begin{cases} (x')g & \text{if } x \in ran\ f \text{ and } (x')f = x, \\ y' & \text{otherwise.} \end{cases}$$

Let $x \in Y$. If $x \notin ran\ f$, then $(x)h = y'$. If $x \in ran\ f$, then $x' \in Y$ by the assumption. This implies that $(x')g \in Y$. Then $h \in M$ and for all $x \in X$, and so $(x)fh = ((x)f)h = (x)g$. Thus $fh = g$ and this implies that $fM = V(X, Y)$. Therefore, $f$ is a left magnifying element in $V(X, Y)$. □

**Example 4.** Consider $X = \mathbb{N}$ and $Y = 2\mathbb{N}$. Let $f \in V(X, Y)$ by $(x)f = x + 4$ for all positive integers $x$, that is,

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & \cdots \end{pmatrix}.$$

Then $dom\ f = X, f$ is one-to-one but not onto a set $X$ and is such that $(y)f^{-1} \subseteq Y$ for all $y \in Y \cap ran\ f$. Let $M = \{h \in V(X, Y) | (1)h = (2)h = (3)h = (4)h = 2\}$. Let $g \in V(X, Y)$ be any function. From Lemma 19, we can define a function $h \in V(X, Y)$ by $(1)h = (2)h = (3)h = (4)h = 2$ and $(x)h = (x - 4)g$ for all $x > 4$. So $h \in M$ and for all $x \in X$, we have

$$(x)fh = ((x)f)h = (x + 4)h = (x)g.$$

For instance, if $g \in V(X, Y)$ is such that $(x)g = 2x$ for all $x \in X$, that is,

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & \cdots \end{pmatrix}.$$

Define a function $h \in V(X, Y)$ by $(1)h = (2)h = (3)h = (4)h = 2$ and $(x)h = (x - 4)g = 2(x - 4)$ for all $x > 4$, that is,

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 2 & 2 & 2 & 2 & 2 & 4 & 6 & \cdots \end{pmatrix}.$$

So $h \in M$ and we obtain

$$\begin{aligned} fh &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & \cdots \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 2 & 2 & 2 & 2 & 2 & 4 & 6 & \cdots \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & \cdots \end{pmatrix} = g. \end{aligned}$$

**Theorem 20.** *$f$ is a left magnifying element in $V(X, Y)$ if and only if $f$ is one-to-one but not onto a set $X$ such that $dom\ f = X$ and $(y)f^{-1} \subseteq Y$ for all $y \in Y \cap ran\ f$.*

*Proof.* Assume that $f$ is a left magnifying element in $V(X, Y)$. By Lemma 15, we gain $dom\ f = X$. By Lemma 16, $f$ is one-to-one. By Lemmas 17 and 18, $f$ is one-to-one but not onto a set $X$ and is such that $(y)f^{-1} \subseteq Y$ for all $y \in Y \cap ran\ f$. Conversely, assume that $f$ is one-to-one but not onto a set $X$, $dom\ f = X$ and $(y)f^{-1} \subseteq Y$ for all $y \in Y \cap ran\ f$. By Lemma 19, we obtain that $f$ is left magnifying in $V(X, Y)$. □

**Corollary 21.** f is a left magnifying element in P(X) if and only if $dom\ f = X$ and f is one-to-one but not onto a set X.

*Proof.* This follows from Theorem 20. □

## 4. Magnifying elements of semigroups of partial transformations preserving partitions

In this section, we find the necessary and sufficient conditions for elements in $S(X, Y)$ to be left and right magnifying.

### 4.1. Right magnifying elements

**Lemma 22.** *If f is a right magnifying element of $S(X, Y)$, then f is onto a set X.*

*Proof.* By the fact that $id_X \in S(X, Y)$, the proof is similar to Lemma 1. □

**Lemma 23.** *Let $f \in S(X, Y)$ be onto a set X.*

1. *If $dom\ f \neq X$, then f is right magnifying.*
2. *If $dom\ f = X$ and f is one-to-one, then f is not right magnifying.*
3. *If $dom\ f = X$ and f is not one-to-one, then f is right magnifying.*

*Proof.* 1. Assume that $dom\ f \neq X$. Let $M = \{h \in S(X, Y) | h$ is not onto a set $X\}$. Then $M \neq S(X, Y)$. Let $g$ be any function in $S(X, Y)$. Since $f$ is onto a set $X$, we can choose for each $x \in dom\ g$ such that $(x)g \in Y, y_x \in Y$ such that $(y_x)f = (x)g$ and for each $x \in dom\ g$ such that $(x)g \in X \setminus Y$, we can choose $y_x \in X \setminus Y$ such that $(y_x)f = (x)g$. Define a function $h \in S(X, Y)$ by $dom\ h = dom\ g$ and $(x)h = y_x$ for all $x \in dom\ g$. Note that if $x \in Y$, then $(x)g \in Y$, and so $y_x \in Y$ and $x \in X \setminus Y$. Thus $(x)g \in X \setminus Y$, and then $y_x \in X \setminus Y$. Since $dom\ f \neq X, h$ is not onto a set $X$. Hence $h \in M$. For all $x \in dom\ g$, we obtain

$$(x)hf = ((x)h)f = (y_x)f = (x)g.$$

Then $hf = g$, and hence $Mf = S(X, Y)$. Therefore, $f$ is right magnifying.

2. Assume that $dom\ f = X$ and $f$ is one-to-one. Then $f$ is bijective. Then its inverse function $f^{-1}$ exists and $f^{-1} \in S(X, Y)$. Suppose that $f$ is a right magnifying element of $S(X, Y)$. Then there exists a proper subset $M$ of $S(X, Y)$ such that $Mf = S(X, Y)$. Hence $Mf = S(X, Y)f$ and $M = Mff^{-1} = S(X, Y)ff^{-1} = S(X, Y)$, a contradiction. Therefore, $f$ is not right magnifying of $S(X, Y)$.

3. Assume that $dom\ f = X$ and $f$ is not one-to-one. Let $M = \{h \in S(X, Y) | h$ is not onto a set $X\}$. Then $M \neq S(X, Y)$. Let $g$ be any function in $S(X, Y)$. Since $f$ is onto a set $X$, we can choose for each $x \in dom\ g$ such that $(x)g \in Y, y_x \in Y$ such that $(y_x)f = (x)g$ (if $(x_1)g = (x_2)g$, we must choose $y_{x_1} = y_{x_2}$) and for each $x \in dom\ g$ such that $(x)g \in X \setminus Y$, we can choose $y_x \in X \setminus Y$ such that $(y_x)f = (x)g$ (if $(x_1)g = (x_2)g$, we must choose $y_{x_1} = y_{x_2}$). Define a function $h \in V(X, Y)$ by

$dom\ h = dom\ g$ and $(x)h = y_x$ for all $x \in dom\ g$. Note that if $x \in Y$, then $(x)g \in Y$, and so $y_x \in Y$ and if $x \in X \setminus Y$, then $y_x \in X \setminus Y$. Next, we claim that $h$ is not onto a set $X$. Since $f$ is not one-to-one, there exist an element $x' \in X$ and distinct elements $x_1, x_2 \in X$ such that $(x_1)f = (x_2)f = x'$. If $x' \notin ran\ g$, we have $x_1, x_2 \notin ran\ h$. If $x' \in ran\ g$, there is only one between $x_1$ and $x_2$ in $ran\ h$. Then $h$ is not onto a set $X$. Hence $h \in M$ and for all $x \in X$, we have $(x)hf = ((x)h)f = (y_x)f = (x)g$. Then $hf = g$, and hence $Mf = S(X, Y)$. Therefore, $f$ is right magnifying. □

**Example 5.** Consider $X = \mathbb{N}$ and $Y = 2\mathbb{N}$.

1. Let $f \in S(X, Y)$ be such that $(2x+1)f = 2x-1$ and $(2x)f = 2x-2$ for all positive integer $x$, that is,

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ - & 2 & 1 & 4 & 3 & 6 & 5 & 8 & \cdots \end{pmatrix}.$$

Then $f$ is onto a set $X$ and $dom\ f \neq X$ since $1 \notin dom\ f$. Let $M = \{h \in S(X, Y) | h \text{ is not onto a set } X\}$. Let $g \in S(X, Y)$ be any function. By Lemma 23(1), there exists $h \in M$ such that $hf = g$. For example, if $g \in S(X, Y)$ such that $(2x)g = 4x$ for all a positive integer $x$, that is,

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ - & 4 & - & 8 & - & 12 & - & 16 & \cdots \end{pmatrix}.$$

Define a function $h \in S(X, Y)$ by $(2x)h = 4x$ for all $x \in X$, that is,

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ - & 4 & - & 8 & - & 12 & - & 16 & \cdots \end{pmatrix}.$$

So $h \in M$ and we have

$$\begin{aligned} hf &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ - & 4 & - & 8 & - & 12 & - & 16 & \cdots \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ - & 2 & 1 & 4 & 3 & 6 & 5 & 8 & \cdots \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ - & 4 & - & 8 & - & 12 & - & 16 & \cdots \end{pmatrix} = g. \end{aligned}$$

2. Let $f \in P(X, Y)$ by $(1)f = 1$ and $(2x)f = 2x$ and $(2x+1)f = 2x-1$ for all positive integer $x$, that is,

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & 1 & 4 & 3 & 6 & 5 & 8 & \cdots \end{pmatrix}.$$

Then $dom\ f = X$, $f$ is onto a set $X$ and $f$ is not one-to-one. Let $M = \{h \in S(X, Y) | h \text{ is not onto a set } X\}$. Let $g \in S(X, Y)$ be any function. By Lemma 23(4), there exists $h \in M$ such that $hf = g$. For example, let $g \in S(X, Y)$ be such that $(x)g = 2x$ for all $x \in X$, that is,

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ - & 4 & - & 8 & - & 12 & - & 16 & \cdots \end{pmatrix}.$$

Define a function $h \in P(X, Y)$ by $(x)h = 2x$ for all positive integer $x$, that is,

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ - & 4 & - & 8 & - & 12 & - & 16 & \cdots \end{pmatrix}.$$

So $h \in M$ and we obtain

$$hf = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ - & 4 & - & 8 & - & 12 & - & 16 & \cdots \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & 1 & 4 & 3 & 6 & 5 & 8 & \cdots \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ - & 4 & - & 8 & - & 12 & - & 16 & \cdots \end{pmatrix} = g.$$

**Theorem 24.** *Let $f \in S(X, Y)$. Then $f$ is right magnifying in S(X, Y) if and only if $f$ is onto a set $X$ and is such that either*

1. *$dom\ f \neq X$ or*
2. *$dom\ f = X$ and $f$ is not one-to-one.*

*Now, we have that the conditions for elements in S(X, Y) to be right magnifying are different from the conditions for elements in P(X, Y) and V(X, Y) to be right magnifying.*

### 4.2. Left magnifying elements

**Lemma 25.** *If $f$ is a left magnifying element of S(X, Y), then $dom\ f = X$.*

*Proof.* Let $f$ be a left magnifying element in S(X, Y). So there exists a proper subset $M$ of S(X, Y) such that $fM = S(X, Y)$. Since $id_X \in S(X, Y)$, there exists $h \in M$ such that $fh = id_X$. This implies that $dom\ f = X$. □

**Lemma 26.** *If $f$ is a left magnifying element in S(X, Y), then $f$ is one-to-one.*

*Proof.* Assume that $f$ is a left magnifying element in $S(X, Y)$. Then there exists a proper subset $M$ of S(X, Y) such that $fM = S(X, Y)$. Since $id_X \in S(X, Y)$, there exists $g \in M$ such that $fg = id_X$. This implies that $f$ is one-to-one. □

**Lemma 27.** *Let $f \in S(X, Y)$ be such that $dom\ f = X$. If $f$ is bijective, then $f$ is not a left magnifying element in S(X, Y).*

*Proof.* Assume that $f$ is left magnifying. Then there exists a proper subset $M$ of S(X, Y) such that $fM = S(X, Y)$. Then $fM = fS(X, Y)$. By assumption, $f^{-1} \in S(X, Y)$. So $M = f^{-1}fM = f^{-1}fS(X, Y) = S(X, Y)$, a contradiction. Hence $f$ is not a left magnifying element in S(X, Y). □

**Lemma 28.** *Let $f \in S(X, Y)$ be such that $dom\ f = X$. If $f$ is one-to-one but not onto a set $X$, then $f$ is a left magnifying element in V(X, Y).*

*Proof.* Assume that $f$ is one-to-one but not onto a set $X$. Let $M = \{h \in V(X, Y) | (x)h = x$ for all $x \notin ran\ f\}$. Then $M \neq S(X, Y)$. We claim that $fM = S(X, Y)$. Let $g \in S(X, Y)$. Define a function $h \in S(X, Y)$ by for all $x \in X$,

$$(x)h = \begin{cases} (x')g & \text{if } x \in ran\ f \text{ and } (x')f = x, \\ x & \text{otherwise.} \end{cases}$$

Let $x \in Y$. If $x \notin ran\ f$, then $(x)h = y'$. If $x \in ran\ f$, then $x' \in Y$ by assumption, and this implies that $(x')g \in Y$. Similarly, if $x \in X \setminus Y$, then $x' \in X \setminus Y$. Thus $h \in M$ and for all $x \in X$, we gain that $(x)fh = ((x)f)h = (x)g$. Hence $fh = g$, and so $fM = S(X, Y)$. Therefore, $f$ is a left magnifying element in $S(X, Y)$. □

**Example 6.** Consider $X = \mathbb{N}$ and $Y = 2\mathbb{N}$. Let $f \in V(X, Y)$ by $(x)f = x + 4$ for all positive integers $x$, that is,

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & \cdots \end{pmatrix}.$$

Then $dom\ f = X, f$ is one-to-one but not onto a set $X$. Let $M = \{h \in V(X, Y) | (1)h = 1, (2)h = 2, (3)h = 3, (4)h = 4\}$. Let $g \in S(X, Y)$ be any function. From Lemma 28, we can define a function $h \in S(X, Y)$ by $(1)h = 1, (2)h = 2, (3)h = 3, (4)h = 4$ and $(x)h = (x - 4)g$ for all $x > 4$. So $h \in M$ and for all $x \in X$, we have

$$(x)fh = ((x)f)h = (x + 4)h = (x)g.$$

For instance, if $g \in S(X, Y)$ such that $(2x)g = 4x$ for all $x \in X$, that is,

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ - & 4 & - & 8 & - & 12 & - & \cdots \end{pmatrix}.$$

Define a function $h \in V(X, Y)$ by $(1)h = 1, (2)h = 2, (3)h = 3, (4)h = 4$ and $(2x)h = (2x - 4)g = 2(2x - 4)$ for all $x > 2$, that is,

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 1 & 2 & 3 & 4 & - & 4 & - & \cdots \end{pmatrix}.$$

So $h \in M$ and we have

$$\begin{aligned} fh &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & \cdots \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 1 & 2 & 3 & 4 & - & 4 & - & \cdots \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ - & 4 & - & 8 & - & 12 & - & \cdots \end{pmatrix} = g. \end{aligned}$$

**Theorem 29.** *$f$ is a left magnifying element in $S(X, Y)$ if and only if $dom\ f = X$ and $f$ is one-to-one but not onto a set $X$.*

*Proof.* This follows from Lemmas 25, 26, 27, and 28. □

**Corollary 30.** *A function $f$ is a left magnifying element in $P(X)$ if and only if $dom\ f = X$ and $f$ is one-to-one but not onto a set $X$.*

*Proof.* This follows from Theorem 29. □

## 5. Conclusion

As a result of Section 2, we investigated the necessary and sufficient conditions for elements in $P(X, Y)$ to be left or right magnifying.

1. $f$ is right magnifying in $P(X, Y)$ if and only if $f$ is onto a set $Y$ such that either
   $Y \not\subseteq dom\ f$ and $(y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ or
   $Y \subseteq dom\ f, (y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ and $|(y)f^{-1} \cap Y| > 1$ for some $y \in Y$.

2. If $|Y| < |X|$, then $P(X, Y)$ has no left magnifying element.
3. If $|Y| = |X|$ and $Y \neq X$, then $f$ is left magnifying of $P(X, Y)$ if and only if $dom\ f = X$ and $f$ is one-to-one.

In Section 3, we give necessary and sufficient conditions for elements in $V(X, Y)$ to be left or right magnifying.

1. $f$ is right magnifying in $V(X, Y)$ if and only if $f$ is onto a set $X$ such that either
   $Y \nsubseteq dom\ f$ and $(y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ or
   $Y \subseteq dom\ f, (y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ and $f$ is not one-to-one.
2. $f$ is left magnifying if and only if $f$ is one-to-one but not onto a set $X$ such that $dom\ f = X$ and $(y)f^{-1} \subseteq Y$ for all $y \in Y \cap ran\ f$.

The necessary and sufficient conditions for elements in $S(X, Y)$ to be left or right magnifying are investigated in Section 4.

1. $f$ is right magnifying in $P(X)$ if and only if $f$ is onto a set $X$ and is such that either
   $dom\ f \neq X$ or
   $dom\ f = X$ and $f$ is not one-to-one.
2. $f$ is left magnifying in $S(X, Y)$ if and only if $dom\ f = X$ and $f$ is one-to-one but not onto a set $X$.

The necessary and sufficient conditions for elements in $P(X)$ to be left or right magnifying are given in Sections 2, 3, and 4.

1. $f$ is right magnifying in $P(X)$ if and only if $f$ is onto a set $X$ and is such that either
   $dom\ f \neq X$ or
   $dom\ f = X$ and $f$ is not one-to-one.
2. $f$ is left magnifying in $P(X)$ if and only if $dom\ f = X$ and $f$ is one-to-one but not onto a set $X$.

## Acknowledgment

This paper was supported by Algebra and Applications Research Unit, Faculty of Science, Prince of Songkla University

## ORCID

Ronnason Chinram http://orcid.org/0000-0002-6113-3689

## References

[1] Catino, F., Migliorini, F. (1992). Magnifying elements in semigroups. *Semigroup Forum* 44(1):314–319. DOI: 10.1007/BF02574350.
[2] Chinram, R., Baupradist, S. (2018). Magnifying elements in a semigroup of transformations with restricted range. *Missouri J. Math. Sci.* 30(1):54–58. DOI: 10.35834/mjms/1534384954.
[3] Chinram, R., Baupradist, S. (2019). Magnifying elements in semigroups of transformations with variant set. *Asian-European J. Math.* 12(04):1950056. DOI: 10.1142/S1793557119500566.
[4] Chinram, R., Petchkaew, P., Baupradist, S. (2018). Left and right magnifying elements in generalized semigroups of transformations by using partitions of a set. *Eur. J. Pure Appl. Math.* 11(3):580–588. DOI: 10.29020/nybg.ejpam.v11i3.3260.

[5] Gutan, M. (1997). Semigroups which contain magnifying elements are factorizable. *Commun. Algebra* 25(12):3953–3963. DOI: 10.1080/00927879708826098.
[6] Gutan, M. (1999). Semigroups with magnifiers admitting minimal subsemigroups. *Commun. Algebra* 27(4): 1975–1996. DOI: 10.1080/00927879908826544.
[7] Gutan, M. (1996). Semigroups with strong and non strong magnifying elements. *Semigroup Forum* 53(1): 384–386. DOI: 10.1007/BF02574152.
[8] Gutan, M., Kisielewicz, A. (2003). Semigroups with good and bad magnifiers. *J. Algebra* 267(2):587–607. DOI: 10.1016/S0021-8693(03)00396-X.
[9] Kaewnoi, T., Petapirak, M., Chinram, R. (2020). Magnifiers in some generalization of the full transformation semigroups. *Mathematics* 8(4):473. DOI: 10.3390/math8040473.
[10] Ljapin, E. S. (1963). *Semigroups, Transl. Math. Monographs* Vol. 3, Providence RI: American Mathematical Society.
[11] Magill, K. D. Jr., (1994). Magnifying elements of transformation semigroups. *Semigroup Forum* 48(1): 119–126. DOI: 10.1007/BF02573659.
[12] Migliorini, F. (1971). Some research on semigroups with magnifying elements. *Period. Math. Hung.* 1(4): 279–286. DOI: 10.1007/BF02020312.
[13] Migliorini, F. (1974). Magnifying elements and minimal subsemigroups in semigroups. *Period. Math. Hung.* 5(4):279–288. DOI: 10.1007/BF02018183.