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ON GENERALIZED FIBONACCI AND k-GENERALIZED FIBONACCI NUMBERS

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Abstract. *In this paper, we provide some new identities for a family of k-generalized Fibonacci numbers which are a generalization of both Fibonacci numbers and Lucas numbers. We investigate the relationships among the terms of k-generalized Fibonacci numbers and examine the sum and the difference of those numbers, especially in case of $kn \pm r$ where $r \in \{0, 1, 2, 3\}$.*
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1. Introduction.

Fibonacci numbers are one of the most well-known numbers in mathematics. Also, those numbers have many important applications to various fields [1], e.g., computer science, physics, biology, and statistics. Some applications of Fibonacci sequences in group theory were studied by Campbell et al., see in [2, 3, 4]. The Fibonacci numbers F_n are given by the recurrence relation:

$$F_{n+1} = F_n + F_{n-1}, \quad n \geq 1$$

with the initial values $F_0 = 0$, $F_1 = 1$.

Koshy [5] wrote one of the most popular books of Fibonacci and Lucas numbers. Those books have many applications of Fibonacci numbers to various fields of mathematics and science and numerous recurrence relations, which are a generalization of Fibonacci and Lucas numbers. For $a, b \in \mathbb{R}$ and $n \geq 1$, the generalized Fibonacci numbers are defined by

$$G_{n+1} = G_n + G_{n-1}$$

with the initial values $G_0 = a$, $G_1 = b$.

El-Mikkawy and Sogabe, see in [6], gave the definition of generalized k-Fibonacci numbers as follows:

$$F_n^{(k)} = (F_m)^{k-r}(F_{m+1})^r, \quad n = mk + r \ (0 \leq r < k),\ m, r \in \mathbb{N} \cup \{0\}.$$

It can be seen that there are several studies of them in the same manner as Fibonacci numbers. The Fibonacci, generalized Fibonacci and generalized k-Fibonacci numbers were inverstigated by many researchers [7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. However, almost identities obtained from the study of k-Fibonacci and generalized k-Fibonacci numbers are the cases $k = 2$ and $k = 3$.

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In this paper, we investigate k-generalized Fibonacci numbers where k is an arbitrary positive integer. We provide some identities and relationships among those numbers, especially in the case of $kn \pm r$ where $r \in \{0, 1, 2, 3\}$. In the results, we focus on the sum and the difference of those numbers.

2. **Preliminaries.** Throughout this research, we use the definition of k-generalized Fibonacci number, which was introduced by Yilmaz et al., see in [17]. For n, k $(k \neq 0) \in \mathbb{N}$ the k-generalized Fibonacci numbers are defined by

$$\begin{aligned} G_n^{(k)} &= \frac{1}{\left(\sqrt{5}\right)^k} \left([a+b\alpha]\alpha^{m-1} - [a+b\beta]\beta^{m-1}\right)^{k-r} \left([a+b\alpha]\alpha^{m} - [a+b\beta]\beta^{m}\right)^{r}; \\ n &= mk + r\,(0 \leq r < k),\ m, r \in \mathbb{N} \cup \{0\} \end{aligned}$$

where $\alpha = \left(1+\sqrt{5}\right)/\,2$ and $\beta = \left(1-\sqrt{5}\right)/\,2$.

Moreover, they gave a relationship of between k-generalized Fibonacci numbers and generalized Fibonacci numbers as follows:

$$G_n^{(k)} = (G_m)^{k-r}(G_{m+1})^r, \quad n = mk + r\ (0 \leq r < k),\ m, r \in \mathbb{N} \cup \{0\}.$$

From the above equation, we have that the 1-generalized Fibonacci number $G_n^{(1)}$ is just the ordinary generalized Fibonacci number G_n because $G_n^{(1)}$ is $a, b, a+b, a+2b, 2a+3b, 3a+5b, 5a+8b, 8a+13b, 13a+21b, 21a+34b, 34a+55b, \ldots$ The first few numbers of the k-generalized Fibonacci numbers for $k = 2, 3$ are as follows:

$$\begin{aligned} \left\{G_n^{(2)}\right\}_{n=0}^{10} &= \left\{a^2, ab, b^2, ab+b^2, a^2+2ab+b^2, a^2+3ab+2b^2, a^2+4ab+ab^2,\right. \\ &\quad \left. 2a^2+7ab+6b^2, 4a^2+12ab+9b^2, 6a^2+19ab+15b^2, 9a^2+30ab+25b^2\right\} \\ \left\{G_n^{(3)}\right\}_{n=0}^{10} &= \left\{a^3, a^2b, ab^2, b^3, ab^2+b^3, a^2b+2ab^2+b^3, a^3+3a^2b+3ab^2+b^3,\right. \\ &\quad a^3+4a^2b+5ab^2+2b^3, a^3+5a^2b+8ab^2+4b^3, a^3+6a^2b+12ab^2+8b^3, \\ &\quad \left. 2a^3+11a^2b+20ab^2+12b^3\right\}. \end{aligned}$$

3. **Main Results.** In this section, we provide some relationships among the terms of k-generalized Fibonacci numbers and the relationships among generalized Fibonacci number and k-generalization Fibonacci numbers, especially in the case of $kn \pm r$ where $r \in \{0, 1, 2, 3\}$.

Theorem 3.1. *Let n, k be positive integers. Then*

$$G_{kn+1}^{(k)} = G_{kn}^{(k)} + G_{kn-1}^{(k)}.$$

Proof: We have $G_n^{(k)} = (G_m)^{k-r}(G_{m+1})^r$ for $n = mk + r$ $(0 \leq r < k)$. Consider

$$\begin{aligned} G_{kn+1}^{(k)} &= G_n^{k-1}G_{n+1} \\ &= G_n^{k-1}(G_n + G_{n-1}) \\ &= G_{kn}^{(k)} + G_{kn-1}^{(k)}. \end{aligned}$$

□

Corollary 3.1. *Let n be a positive integer. Then*

$$G_{3n+1}^{(3)} = G_{3n}^{(3)} + G_{3n-1}^{(3)}.$$

Proof: For $k = 3$, the corollary follows by Theorem 3.1. □

Theorem 3.2. *Let n, k be positive integers. Then*

$$G_{kn+2}^{(k)} = G_{kn}^{(k)} + 2G_{kn-1}^{(k)} + G_{kn-2}^{(k)}.$$

Proof: Since $G_n^{(k)} = (G_m)^{k-r}(G_{m+1})^r$ for $n = mk + r$ $(0 \le r < k)$, we consider

$$\begin{aligned} G_{kn+2}^{(k)} &= G_n^{k-2} G_{n+1}^2 \\ &= G_n^{k-2}(G_n + G_{n-1})^2 \\ &= G_{kn}^{(k)} + 2G_{kn-1}^{(k)} + G_{kn-2}^{(k)}. \end{aligned}$$

□

Corollary 3.2. *Let n be a positive integer. Then*

$$G_{3n+2}^{(3)} = G_{3n}^{(3)} + 2G_{3n-1}^{(3)} + G_{3n-2}^{(3)}.$$

Proof: For $k = 3$, the corollary follows by Theorem 3.2. □

Theorem 3.3. *Let n, k be positive integers. Then*

$$G_{kn-2}^{(k)} = G_{kn+2}^{(k)} - 2G_{kn+1}^{(k)} + G_{kn}^{(k)}.$$

Proof: We have $G_n^{(k)} = (G_m)^{k-r}(G_{m+1})^r$ for $n = mk + r$ $(0 \le r < k)$. Consider

$$\begin{aligned} G_{kn-2}^{(k)} &= G_{k(n-1)+(k-2)}^{(k)} \\ &= G_{n-1}^{k-(k-2)} G_{(n-1)+1}^{k-2} \\ &= G_n^{k-2} G_{n-1}^2 \\ &= G_n^{k-2}(G_{n+1} - G_n)^2 \\ &= G_{kn+2}^{(k)} - 2G_{kn+1}^{(k)} + G_{kn}^{(k)}. \end{aligned}$$

□

Corollary 3.3. *Let n be a positive integer. Then*

$$G_{3n-2}^{(3)} = G_{3n+2}^{(3)} - 2G_{3n+1}^{(3)} + G_{3n}^{(3)}.$$

Proof: For $k = 3$, the corollary follows by Theorem 3.3. □

Theorem 3.4. *Let n, k be positive integers. Then*

$$G_{kn+2}^{(k)} - G_{kn-2}^{(k)} = G_{kn-1}^{(k)} + G_{kn+1}^{(k)}.$$

Proof: We have $G_n^{(k)} = (G_m)^{k-r}(G_{m+1})^r$ for $n = mk + r$ $(0 \le r < k)$. By Theorem 3.2 and Theorem 3.3, we get

$$\begin{aligned} G_{kn+2}^{(k)} - G_{kn-2}^{(k)} &= \left(G_{kn}^{(k)} + 2G_{kn-1}^{(k)} + G_{kn-2}^{(k)}\right) - \left(G_{kn+2}^{(k)} - 2G_{kn+1}^{(k)} + G_{kn}^{(k)}\right) \\ &= 2G_{kn-1}^{(k)} + G_{kn-2}^{(k)} - G_{kn+2}^{(k)} + 2G_{kn+1}^{(k)}. \end{aligned}$$

Then, we have

$$\begin{aligned} & G_{kn+2}^{(k)} - G_{kn-2}^{(k)} - G_{kn-2}^{(k)} + G_{kn+2}^{(k)} = 2G_{kn-1}^{(k)} + 2G_{kn+1}^{(k)} \\ & 2\left(G_{kn+2}^{(k)} - G_{kn-2}^{(k)}\right) = 2\left(G_{kn-1}^{(k)} + 2G_{kn+1}^{(k)}\right) \\ & G_{kn+2}^{(k)} - G_{kn-2}^{(k)} = G_{kn-1}^{(k)} + G_{kn+1}^{(k)}. \end{aligned}$$

□

Corollary 3.4. *Let n be a positive integer. Then*

$$G_{3n+2}^{(3)} - G_{3n-2}^{(3)} = G_{3n-1}^{(3)} + G_{3n+1}^{(3)}.$$

Proof: For $k = 3$, the corollary follows by Theorem 3.4. □

Theorem 3.5. *Let n, k be positive integers. Then*

$$G_{kn+3}^{(k)} = G_{kn}^{(k)} + 3G_{kn-1}^{(k)} + 3G_{kn-2}^{(k)} + G_{kn-3}^{(k)}.$$

Proof: We have $G_n^{(k)} = (G_m)^{k-r}(G_{m+1})^r$ for $n = mk + r$ $(0 \leq r < k)$. Consider

$$\begin{aligned} G_{kn+3}^{(k)} &= G_n^{k-3} G_{n+1}^3 \\ &= G_n^{k-3}(G_n + G_{n-1})^3 \\ &= G_{kn}^{(k)} + 3G_{kn-1}^{(k)} + 3G_{kn-2}^{(k)} + G_{kn-3}^{(k)}. \end{aligned}$$ □

Corollary 3.5. *Let n be a positive integer. Then*

$$G_{3n+3}^{(3)} = G_{3n}^{(3)} + 3G_{3n-1}^{(3)} + 3G_{3n-2}^{(3)} + G_{3n-3}^{(3)}.$$

Proof: For $k = 3$, the corollary follows by Theorem 3.5. □

Theorem 3.6. *Let n, k be positive integers. Then*

$$G_{kn-3}^{(k)} = G_{kn+3}^{(k)} - 3G_{kn+2}^{(k)} + 3G_{kn+1}^{(k)} - G_{kn}^{(k)}.$$

Proof: Since $G_n^{(k)} = (G_m)^{k-r}(G_{m+1})^r$ for $n = mk + r$ $(0 \leq r < k)$, we consider

$$\begin{aligned} G_{kn-3}^{(k)} &= G_{k(n-1)+(k-3)}^{(k)} \\ &= G_{n-1}^{k-(k-3)} G_{(n-1)+1}^{k-3} \\ &= G_{n-1}^3 G_n^{k-3} \\ &= G_n^{k-3}(G_{n+1} - G_n)^3 \\ &= G_{kn+3}^{(k)} - 3G_{kn+2}^{(k)} + 3G_{kn+1}^{(k)} - G_{kn}^{(k)}. \end{aligned}$$ □

Corollary 3.6. *Let n be a positive integer. Then*

$$G_{3n-3}^{(3)} = G_{3n+3}^{(3)} - 3G_{3n+2}^{(3)} + 3G_{3n+1}^{(3)} - G_{3n}^{(3)}.$$

Proof: For $k = 3$, the corollary follows by Theorem 3.6. □

Theorem 3.7. *Let n, k be positive integers. Then*

$$G_{kn+3}^{(k)} - G_{kn-3}^{(k)} = 4G_{kn}^{(k)}.$$

Proof: Since $G_n^{(k)} = (G_m)^{k-r}(G_{m+1})^r$ for $n = mk + r$ $(0 \leq r < k)$, we consider

$$\begin{aligned} & G_{kn+3}^{(k)} - G_{kn-3}^{(k)} \\ &= \left(G_{kn}^{(k)} + 3G_{kn-1}^{(k)} + 3G_{kn-2}^{(k)} + G_{kn-3}^{(k)}\right) - \left(G_{kn+3}^{(k)} - 3G_{kn+2}^{(k)} + 3G_{kn+1}^{(k)} - G_{kn}^{(k)}\right) \\ &= 2G_{kn}^{(k)} + 3G_{kn-1}^{(k)} + 3G_{kn-2}^{(k)} + G_{kn-3}^{(k)} - G_{kn+3}^{(k)} + 3G_{kn+2}^{(k)} - 3G_{kn+1}^{(k)}. \end{aligned}$$

Then, we have

$$G_{kn+3}^{(k)} - G_{kn-3}^{(k)} - G_{kn-3}^{(k)} + G_{kn+3}^{(k)} = 2G_{kn}^{(k)} + 3G_{kn-1}^{(k)} + 3G_{kn-2}^{(k)} + 3G_{kn+2}^{(k)} - 3G_{kn+1}^{(k)}.$$

By Theorem 3.5 and Theorem 3.6, we obtain $G_{kn-1}^{(k)} + G_{kn-2}^{(k)} = G_{kn}^{(k)}$ and $G_{kn+2}^{(k)} - G_{kn+1}^{(k)} = G_{kn}^{(k)}$. So, we get

$$\begin{aligned} 2G_{kn+3}^{(k)} - 2G_{kn-3}^{(k)} &= 2G_{kn}^{(k)} + 3G_{kn}^{(k)} + 3G_{kn}^{(k)} \\ 2\left(G_{kn+3}^{(k)} - G_{kn-3}^{(k)}\right) &= 8G_{kn}^{(k)} \\ G_{kn+3}^{(k)} - G_{kn-3}^{(k)} &= 4G_{kn}^{(k)}. \end{aligned}$$ □

Corollary 3.7. *Let n be a positive integer. Then*

$$G_{3n+3}^{(3)} - G_{3n-3}^{(3)} = 4G_{3n}^{(3)}.$$

Proof: For $k = 3$, the corollary follows by Theorem 3.7. □

Theorem 3.8. *Let n, k be positive integers. Then*

$$G_{kn+3}^{(k)} + G_{kn}^{(k)} = G_{kn+2}^{(k)} + 2G_{kn-1}^{(k)} + 3G_{kn-2}^{(k)} + G_{kn-3}^{(k)}.$$

Proof: We have $G_n^{(k)} = (G_m)^{k-r}(G_{m+1})^r$ for $n = mk + r$ $(0 \le r < k)$. By Theorem 3.1 and Theorem 3.5, we consider

$$\begin{aligned} G_{kn+3}^{(k)} + G_{kn}^{(k)} &= \left(G_{kn}^{(k)} + 3G_{kn-1}^{(k)} + 3G_{kn-2}^{(k)} + G_{kn-3}^{(k)}\right) + \left(G_{kn+1}^{(k)} - G_{kn-1}^{(k)}\right) \\ &= G_{kn+2}^{(k)} + 2G_{kn-1}^{(k)} + 3G_{kn-2}^{(k)} + G_{kn-3}^{(k)}. \end{aligned}$$ □

Corollary 3.8. *Let n be a positive integer. Then*

$$G_{3n+3}^{(3)} + G_{3n}^{(3)} = G_{3n+2}^{(3)} + 2G_{3n-1}^{(3)} + 3G_{3n-2}^{(3)} + G_{3n-3}^{(3)}.$$

Proof: For $k = 3$, the corollary follows by Theorem 3.8. □

Theorem 3.9. *Let n, k be positive integers. Then*

$$G_{kn+3}^{(k)} - G_{kn}^{(k)} = 3G_{kn-1}^{(k)} + 3G_{kn-2}^{(k)} + G_{kn-3}^{(k)}.$$

Proof: We have $G_n^{(k)} = (G_m)^{k-r}(G_{m+1})^r$ for $n = mk + r$ $(0 \le r < k)$. By Theorem 3.1 and Theorem 3.5, we consider

$$\begin{aligned} G_{kn+3}^{(k)} - G_{kn}^{(k)} &= \left(G_{kn}^{(k)} + 3G_{kn-1}^{(k)} + 3G_{kn-2}^{(k)} + G_{kn-3}^{(k)}\right) - \left(G_{kn+1}^{(k)} - G_{kn-1}^{(k)}\right) \\ &= 3G_{kn-1}^{(k)} + 3G_{kn-2}^{(k)} + G_{kn-3}^{(k)}. \end{aligned}$$ □

Corollary 3.9. *Let n be a positive integer. Then*

$$G_{3n+3}^{(3)} - G_{3n}^{(3)} = 3G_{3n-1}^{(3)} + 3G_{3n-2}^{(3)} + G_{3n-3}^{(3)}.$$

Proof: For $k = 3$, the corollary follows by Theorem 3.9. □

Theorem 3.10. *Let n, k be positive integers. Then*

$$G_{kn-3}^{(k)} + G_{kn-2}^{(k)} + G_{kn-1}^{(k)} = G_{kn+3}^{(k)} - 2G_{kn+2}^{(k)} + 2G_{kn+1}^{(k)} - G_{kn}^{(k)}.$$

Proof: We have $G_n^{(k)} = (G_m)^{k-r}(G_{m+1})^r$ for $n = mk + r$ $(0 \le r < k)$. By Theorem 3.1, Theorem 3.3 and Theorem 3.6, we obtain

$$\begin{aligned} &G_{kn-3}^{(k)} + G_{kn-2}^{(k)} + G_{kn-1}^{(k)} \\ &= \left(G_{kn+3}^{(k)} - 3G_{kn+2}^{(k)} + 3G_{kn+1}^{(k)} - G_{kn}^{(k)}\right) + \left(G_{kn+2}^{(k)} - 2G_{kn+1}^{(k)} + G_{kn}^{(k)}\right) + \left(G_{kn+1}^{(k)} - G_{kn}^{(k)}\right) \\ &= G_{kn+3}^{(k)} - 2G_{kn+2}^{(k)} + 2G_{kn+1}^{(k)} - G_{kn}^{(k)}. \end{aligned}$$ □

Corollary 3.10. *Let n be a positive integer. Then*

$$G_{3n-3}^{(3)} + G_{3n-2}^{(3)} + G_{3n-1}^{(3)} = G_{3n+3}^{(3)} - 2G_{3n+2}^{(3)} + 2G_{3n+1}^{(3)} - G_{3n}^{(3)}.$$

Proof: For $k = 3$, the corollary follows by Theorem 3.10. □

Example 3.1. *From Corollary 3.10, if we consider in case $n = 1$, we have*

$$\begin{aligned} &G_6^{(3)} - 2G_5^{(3)} + 2G_4^{(3)} - G_2^{(3)} \\ &= a^3 + 3a^2b + 3ab^2 + b^3 - 2\left(a^2b + 2ab^2 + b^3\right) + 2\left(ab^2 + b^3\right) - \left(ab^2 + b^3\right) \\ &= a^3 + a^2b + ab^2 \\ &= G_0^{(3)} + G_1^{(3)} + G_2^{(3)}. \end{aligned}$$

Theorem 3.11. *Let n be a positive integer. Then*

$$G_n(G_{n-1})^2 = G_{3n+2}^{(3)} - 2G_{3n+1}^{(3)} + G_{3n}^{(3)}.$$

Proof: We have $G_n^{(k)} = (G_m)^{k-r}(G_{m+1})^r$ for $n = mk + r$ $(0 \leq r < k)$. Consider

$$\begin{aligned} G_n(G_{n-1})^2 &= G_n(G_{n+1} - G_n)^2 \\ &= G_n G_{n+1}^2 - 2G_n^2 G_{n+1} + G_n^3 \\ &= G_{3n+2}^{(3)} - 2G_{3n+1}^{(3)} + G_{3n}^{(3)}. \end{aligned}$$ □

Theorem 3.12. *Let n be a positive integer. We have*

$$G_{n-1}(G_n)^2 = G_{3n+3}^{(3)} - G_{3n+2}^{(3)} - 2G_{3n-2}^{(3)} - G_{3n-3}^{(3)}.$$

Proof: Since $G_n^{(k)} = (G_m)^{k-r}(G_{m+1})^r$ for $n = mk + r$ $(0 \leq r < k)$, we consider

$$\begin{aligned} G_{n-1}(G_n)^2 &= G_{n-1}(G_{n+1} - G_{n-1})^2 \\ &= G_{n-1}G_{n+1}^2 - 2G_{n-1}^2 G_{n+1} + G_{n-1}^3 \\ &= G_{n+1}^2(G_{n+1} - G_n) - 2G_{n-1}^2(G_n + G_{n-1}) + G_{n-1}^3 \\ &= G_{n+1}^3 - G_n G_{n+1}^2 - 2G_{n-1}^2 G_n - G_{n-1}^3 \\ &= G_{3n+3}^{(3)} - G_{3n+2}^{(3)} - 2G_{3n-2}^{(3)} - G_{3n-3}^{(3)}. \end{aligned}$$ □

Theorem 3.13. *Let n be a positive integer. We have*

$$G_n(G_{n+1})^2 = G_{3n}^{(3)} + 2G_{3n-1}^{(3)} + G_{3n-2}^{(3)}.$$

Proof: Since $G_n^{(k)} = (G_m)^{k-r}(G_{m+1})^r$ for $n = mk + r$ $(0 \leq r < k)$, we obtain

$$\begin{aligned} G_n(G_{n+1})^2 &= G_n(G_n + G_{n-1})^2 \\ &= G_n^3 + 2G_{n-1}G_n^2 + G_{n-1}^2 G_n \\ &= G_{3n}^{(3)} + 2G_{3n-1}^{(3)} + G_{3n-2}^{(3)}. \end{aligned}$$ □

Theorem 3.14. *Let n be a positive integer. We have*

$$G_{n+1}(G_n)^2 = -G_{3n+3}^{(3)} + 2G_{3n+2}^{(3)} + G_{3n-2}^{(3)} + G_{3n-3}^{(3)}.$$

Proof: We have $G_n^{(k)} = (G_m)^{k-r}(G_{m+1})^r$ for $n = mk + r$ $(0 \leq r < k)$. Then

$$\begin{aligned} G_{n+1}(G_n)^2 &= G_{n+1}(G_{n+1} - G_{n-1})^2 \\ &= G_{n+1}^3 - 2G_{n+1}^2 G_{n-1} + G_{n-1}^2 G_{n+1} \\ &= G_{n+1}^3 - 2G_{n+1}^2(G_{n+1} - G_n) + G_{n-1}^2(G_n + G_{n-1}) \\ &= -G_{n+1}^3 + 2G_n G_{n+1}^2 + G_{n-1}^2 G_n + G_{n-1}^3 \\ &= -G_{3n+3}^{(3)} + 2G_{3n+2}^{(3)} + G_{3n-2}^{(3)} + G_{3n-3}^{(3)}. \end{aligned}$$ □

Theorem 3.15. *Let n be a positive integer. We have*

$$(G_{n+1})^3 = G_{3n}^{(3)} + 3G_{3n-1}^{(3)} + 3G_{3n-2}^{3} + G_{3n-3}^{(3)}.$$

Proof: Since $G_n^{(k)} = (G_m)^{k-r}(G_{m+1})^r$ for $n = mk + r$ $(0 \leq r < k)$, we consider

$$\begin{aligned} (G_{n+1})^3 &= (G_n + G_{n-1})^3 \\ &= G_n^3 + 3G_{n-1}G_n^2 + 3G_{n-1}^2 G_n + G_{n-1}^3 \\ &= G_{3n}^{(3)} + 3G_{3n-1}^{(3)} + 3G_{3n-2}^{(3)} + G_{3n-3}^{(3)}. \end{aligned}$$ □

Theorem 3.16. *Let n be a positive integer. We have*

$$(G_{n+2})^3 = G_{3n}^{(3)} + 3G_{3n+1}^{(3)} + 3G_{3n+2}^{(3)} + G_{3n+3}^{(3)}.$$

Proof: Since $G_n^{(k)} = (G_m)^{k-r}(G_{m+1})^r$ for $n = mk + r$ $(0 \le r < k)$, we obtain

$$\begin{aligned}(G_{n+2})^3 &= (G_n + G_{n+1})^3 \\ &= G_n^3 + 3G_n^2G_{n+1} + 3G_nG_{n+1}^2 + G_{n+1}^3 \\ &= G_{3n}^{(3)} + 3G_{3n+1}^{(3)} + 3G_{3n+2}^{(3)} + G_{3n+3}^{(3)}.\end{aligned}$$

□

Theorem 3.17. *Let n be a positive integer. Then*

$$(G_{n+3})^3 = G_{3n+3}^{(3)} + 3G_{3n+4}^{(3)} + 3G_{3n+5}^{(3)} + G_{3n+6}^{(3)}.$$

Proof: We have $G_n^{(k)} = (G_m)^{k-r}(G_{m+1})^r$ for $n = mk + r$ $(0 \le r < k)$. We obtain

$$\begin{aligned}(G_{n+3})^3 &= (G_{n+1} + G_{n+2})^3 \\ &= G_{n+1}^3 + 3G_{n+1}^2G_{n+2} + 3G_{n+1}G_{n+2}^2 + G_{n+2}^3 \\ &= G_{3n+3}^{(3)} + 3G_{3n+4}^{(3)} + 3G_{3n+5}^{(3)} + G_{3n+6}^{(3)}.\end{aligned}$$

□

Theorem 3.18. *Let k, n, t be positive integers. Then*

$$G_{3n+k+t}^{(3)}G_{3n+k+t-2}^{(3)} - \left(G_{3n+k+t-1}^{(3)}\right)^2 = \begin{cases} G_{4n}^{(4)}G_{(k+1)(n+1)}^{(k+1)}G_{(1-k)(n-1)}^{(1-k)} - G_{6n+2k}^{(6)}, & t = 1 \\ 0, & t \ne 1 \end{cases}$$

Proof: For $t = 1$, we obtain

$$\begin{aligned}G_{3n+k+1}^{(3)}G_{3n+k-1}^{(3)} - \left(G_{3n+k}^{(3)}\right)^2 &= \left(G_{3(n)+(k+1)}^{(3)}G_{3(n-1)+(k+2)}^{(3)}\right) - \left(G_{3(n)+k}^{(3)}\right)^2 \\ &= \left(G_n^{3-k-1}G_{n+1}^{k+1}\right)\left(G_{n-1}^{3-k-2}G_n^{k+2}\right) - \left(G_n^{3-k}G_{n+1}^k\right)^2 \\ &= \left(G_n^4G_{n+1}^{k+1}G_{n-1}^{1-k}\right) - \left(G_n^{6-2k}G_{n+1}^{2k}\right) \\ &= G_{4n}^{(4)}G_{(k+1)(n+1)}^{(k+1)}G_{(1-k)(n-1)}^{(1-k)} - G_{6n+2k}^{(6)}.\end{aligned}$$

For $t \ne 1$, we obtain

$$\begin{aligned}&G_{3n+k+t}^{(3)}G_{3n+k+t-2}^{(3)} - \left(G_{3n+k+t-1}^{(3)}\right)^2 \\ &= G_{3(n)+(k+t)}^{(3)}G_{3(n)+(k+t-2)}^{(3)} - \left(G_{3(n)+(k+t-1)}^{(3)}\right)^2 \\ &= \left(G_n^{3-k-t}G_{n+1}^{k+t}\right)\left(G_n^{3-k-t+2}G_{n+1}^{k+t-2}\right) - \left(G_n^{3-k-t+1}G_{n+1}^{k+t-1}\right)^2 \\ &= \left(G_n^{8-2k-2t}G_{n+1}^{2k+2t-2}\right) - \left(G_n^{8-2k-2t}G_{n+1}^{2k+2t-2}\right) \\ &= 0.\end{aligned}$$

□

Theorem 3.19. *Let n be a positive integer. Then*

$$G_{kn}^{(k)}G_{kn-2}^{(k)} - \left(G_{kn-1}^{(k)}\right)^2 = 0.$$

Proof: We have $G_n^{(k)} = (G_m)^{k-r}(G_{m+1})^r$ for $n = mk + r$ $(0 \le r < k)$. We obtain

$$\begin{aligned}G_{kn}^{(k)}G_{kn-2}^{(k)} - \left(G_{kn-1}^{(k)}\right)^2 &= G_{k(n)+0}^{(k)}G_{k(n-1)+(k-2)}^{(k)} - \left(G_{k(n-1)+(k-1)}^{(k)}\right)^2 \\ &= \left(G_n^k\right)\left(G_{n-1}^2G_n^{k-2}\right) - \left(G_{n-1}G_n^{k-1}\right)^2 \\ &= G_n^{2k-2}G_{n-1}^2 - G_{n-1}^2G_n^{2k-2} \\ &= 0.\end{aligned}$$

□

Theorem 3.20. *Let n be a positive integer. Then*

$$\sum_{i=0}^{k-3}\binom{k-3}{i}G_{kn+3+i}^{(k)} = G_{3n+3}^{(3)}G_{(k-3)(n+2)}^{(k-3)}.$$

Proof: Since $G_n^{(k)} = (G_m)^{k-r}(G_{m+1})^r$ for $n = mk + r$ $(0 \le r < k)$, and the well known binomial property, we obtain

$$\begin{aligned}
\sum_{i=0}^{k-3} \binom{k-3}{i} G_{kn+3+i}^{(k)} &= \sum_{i=0}^{k-3} \binom{k-3}{i} G_{k(n)+(3+i)}^{(k)} \\
&= \sum_{i=0}^{k-3} \binom{k-3}{i} G_n^{k-(3+i)} G_{n+1}^{3+i} \\
&= \sum_{i=0}^{k-3} \binom{k-3}{i} G_n^{k-3-i} G_{n+1}^{3+i} \\
&= G_{n+1}^3 \sum_{i=0}^{k-3} \binom{k-3}{i} G_n^{k-3-i} G_{n+1}^{3+i-3} \\
&= G_{n+1}^3 (G_n + G_{n+1})^{k-3} \\
&= G_{3n+3}^{(3)} G_{(k-3)(n+2)}^{(k-3)}. \qquad \square
\end{aligned}$$

4. **Discussion and Conclusion.** In this paper, we have proved some theorems concerning the k-generalized Fibonacci numbers, especially in the case $kn \pm r$ where $r \in \{0, 1, 2, 3\}$. In Theorem 3.1, Theorem 3.2, and Theorem 3.5, we have written $G_{kn+1}^{(k)}$, $G_{kn+2}^{(k)}$ and $G_{kn+3}^{(k)}$ in terms of $G_{kn-r}^{(k)}$ where $r \in \{0, 1, 2, 3\}$. Mereover, in Theorem 3.3 and Theorem 3.6, we have written $G_{kn-2}^{(k)}$ and $G_{kn-3}^{(k)}$ in terms of $G_{kn+r}^{(k)}$ where $r \in \{0, 1, 2, 3\}$. Moreover, in case of $G_{kn-1}^{(k)}$, it is easy to see in Theorem 3.1. Furthermore, we have given the difference of $G_{kn+r}^{(k)}$ and $G_{kn-r}^{(k)}$ where $r \in \{0, 1, 2, 3\}$ in Theorem 3.4 and Theorem 3.7. Moreover, we express generalized Fibonacci number in the form of k-generalized Fibonacci number distributions in Theorems 3.11-3.17.

In the future, the researchers can investigate the boundary of k-generalized Fibonacci sequences.

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