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ON GENERALIZATION k-FIBONACCI AND k-LUCAS NUMBERS

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Abstract. *In this research, we delve into the generalization of k-Fibonacci and k-Lucas numbers, and rigorously prove the new properties associated with them. Furthermore, we establish constraints on the values of the generalized k-Fibonacci and k-Lucas numbers and investigate relationships between the family of k-Fibonacci numbers and the family of k-Lucas numbers. Additionally, we derive a binomial form that represents the sum of Lucas and Fibonacci numbers in the context of the generalized k-Fibonacci sequences.*
Keywords: Fibonacci number, Lucas number, Generalized k-Fibonacci number, Generalized k-Lucas number

1. **Introduction.** The Fibonacci sequence is a powerful tool that can be used to solve a wide variety of problems. Its applications are still being discovered, and it is likely to find even more uses in the future. The application of Fibonacci sequences in various fields was studied [1, 2, 3]. In 2001, Koshy [4] investigated the application of Fibonacci and Lucas numbers to several fields of mathematics and science. Numerous researchers have extensively studied the generalized Fibonacci and Lucas sequences, as evidenced by their contributions documented in references such as [5, 6, 7, 8, 9, 10, 11, 12]. These sequences were defined in a similar way to the Fibonacci and Lucas sequences, but with different initial values. The concept of generalized k-Fibonacci numbers was first proposed by Mikkawy and Sogabe in 2010 [13]. Similarly, the introduction of k-Lucas numbers is credited to Özkan et al. [14]. Among the intriguing studies on generalized Fibonacci and Lucas numbers, an area of particular interest lies in the investigation of generalized k-Fibonacci and generalized k-Lucas numbers. Noteworthy research on these topics can be

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found in references such as [15, 16, 17]. They explored the relationships between ordinary Fibonacci numbers and generalized k-Fibonacci and the relationships among generalized k-Fibonacci, especially for the cases when $k = 2$ and $k = 3$. Furthermore, they examined the relationship on Lucas and generalized k-Lucas, following a similar approach as they did with Fibonacci numbers and generalized k-Fibonacci numbers.

In this paper, we introduce some properties of generalized k-Fibonacci and generalized k-Lucas numbers. Our investigation focused on establishing constraints on the values of the generalized k-Fibonacci and k-Lucas numbers and exploring the relationships among them, particularly when k is a positive integer. Additionally, we derive a binomial form representing the sum of Lucas and Fibonacci numbers in the context of the generalized k-Fibonacci sequences.

2. **Preliminaries.** The sequence 1, 1, 2, 3, 5, 8, 13, and so on, is widely recognized as the Fibonacci sequence. It is common knowledge that each term of this sequence, denoted as F_n, is determined by the second-order linear recurrence relation $F_n = F_{n-1} + F_{n-2}$, where n takes values starting from 2. The initial values of the sequence are $F_0 = 1$ and $F_1 = 1$.

The Fibonacci numbers can be expressed using Binet's formula, which states that for $n = 0, 1, 2, \ldots$, the nth Fibonacci number, denoted as F_n, is given by the equation:

$$F_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} = \frac{1}{\sqrt{5}} \left(\alpha^{n+1} - \beta^{n+1}\right),$$

where α and β represent the roots of the quadratic equation $x^2 - x - 1 = 0$. Specifically, α is equal to $\frac{1+\sqrt{5}}{2}$, and β is equal to $\frac{1-\sqrt{5}}{2}$.

The Lucas numbers, denoted as L_n, follow a second-order linear recurrence relation given by

$$L_n = L_{n-1} + L_{n-2}, \quad n = 2, 3, 4, \ldots$$

The initial values of the Lucas sequence are $L_0 = 2$ and $L_1 = 1$. Therefore, the Lucas number L_n is a term of the sequence $2, 1, 3, 4, 7, 11, 18, \ldots$. The Lucas numbers can be expressed using Binet's formula as

$$L_n = \alpha^n + \beta^n, \quad n = 0, 1, 2, \ldots$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. There exists a relationship between the Fibonacci numbers and the Lucas numbers, given by

$$L_n = F_n + F_{n-2} = \frac{F_{2n-1}}{F_{n-1}}.$$

The definition of generalized k-Fibonacci numbers was introduced by Mikkawy and Sogabe in 2010 [13].

Definition 2.1. [13] *Let n and k $(\neq 0)$ be natural numbers, and then there exist unique numbers m and r such that $n = mk + r$ $(0 \leq r < k)$. The generalized k-Fibonacci number $F_n^{(k)}$ is defined by*

$$F_n^{(k)} = \frac{1}{\left(\sqrt{5}\right)^k} \left(\alpha^{m+2} - \beta^{m+2}\right)^r \left(\alpha^{m+1} - \beta^{m+1}\right)^{k-r}, \quad n = mk + r$$

where $\alpha = \left(1 + \sqrt{5}\right)/2$ and $\beta = \left(1 - \sqrt{5}\right)/2$.

The relationship between the generalized k-Fibonacci numbers and the Fibonacci numbers can be expressed as

$$F_n^{(k)} = (F_m)^{k-r} \cdot (F_{m+1})^r, \text{ where } n = mk + r.$$

By Definition 2.1, the generalized 1-Fibonacci number $F_n^{(1)}$ corresponds to the ordinary Fibonacci number F_n since the sequence $\left\{F_n^{(1)}\right\} = \{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\}$ remains unchanged. For $k = 2$ and 3, the initial terms of the generalized k-Fibonacci sequences are as follows:

$$\left\{F_n^{(2)}\right\}_{n=0}^{15} = \{1, 1, 1, 2, 4, 6, 9, 15, 25, 40, 64, 104, 169, 273, 441, 714\},$$
$$\left\{F_n^{(3)}\right\}_{n=0}^{15} = \{1, 1, 1, 1, 2, 4, 8, 12, 18, 27, 45, 75, 125, 200, 320, 512\}.$$

The definition of the generalized k-Lucas numbers was presented by Özkan et al. in 2017 [14].

Definition 2.2. [14] *Let n and k $(\neq 0)$ be natural numbers, and then there exist unique numbers m and r such that $n = mk + r$ $(0 \leq r < k)$. The generalized k-Lucas number $L_n^{(k)}$ is defined by*

$$L_n^{(k)} = \left(\alpha^{m+1} + \beta^{m+1}\right)^r (\alpha^m + \beta^m)^{k-r}, \quad n = mk + r$$

where $\alpha = \left(1 + \sqrt{5}\right)/2$ and $\beta = \left(1 - \sqrt{5}\right)/2$.

By Definition 2.2, the relationship between the generalized k-Lucas numbers and the Lucas numbers is given by

$$L_n^{(k)} = (L_m)^{k-r} \cdot (L_{m+1})^r, \text{ where } n = mk + r$$

Analogous to the generalized k-Fibonacci numbers, in the case of $k = 1$, the generalized 1-Lucas number $L_n^{(1)}$ is equivalent to the ordinary Lucas number L_n as the sequence $L_n^{(1)} = \{2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \ldots\}$ remains unaltered.

The initial terms of the generalized k-Lucas numbers for $k = 2$ and 3 are as follows:

$$\left\{L_n^{(2)}\right\}_{n=0}^{15} = \{4, 2, 1, 3, 9, 12, 16, 28, 49, 77, 121, 198, 324, 522, 841, 1363\},$$
$$\left\{L_n^{(3)}\right\}_{n=0}^{15} = \{8, 4, 2, 1, 3, 9, 27, 36, 48, 64, 112, 196, 343, 539, 847, 1331\}.$$

3. **Main Results.** In this section, we investigate the properties of generalized k-Fibonacci and generalized k-Lucas numbers. First, we can generalize each term of the k-Fibonacci sequence in the following theorem.

Theorem 3.1. *Let n and k be positive integers with $k \geq 2$. Then*

$$F_n^{(k)} \begin{cases} = 1 & \text{if } n \leq k \\ = 2^r & \text{if } n = mk + r \text{ and } k < n < 2k \\ = 2^k & \text{if } n = 2k \\ < 2^{n-k} & \text{if } n > 2k. \end{cases}$$

Proof: Case 1: Let $n \leq k$. We divide this case into two subcases.
Subcase 1.1: $n < k$. Then we have $n = 0 \cdot k + n$. Then we obtain $F_n^{(k)} = (F_0)^{k-n}(F_1)^n = 1 \cdot 1 = 1$.
Subcase 1.2: $n = k$. This implies that $F_n^{(k)} = (F_1)^k (F_2)^0 = 1$.
Case 2: Let $k < n < 2k$. Then $n = k + r$ for all $r \in \{1, 2, \ldots, k - 1\}$. This implies that

$$F_n^{(k)} = F_{k+r}^{(k)} = (F_1)^{k-r}(F_2)^r = 1 \cdot 2^r = 2^r.$$

Case 3: Let $n = 2k$. Then we obtain $F_{2k}^{(k)} = (F_2)^k (F_3)^0 = 2^k \cdot 1 = 2^k$.
Case 4: Let $P(n)$ be the given statement, that is, $P(n) : F_n^{(k)} < 2^{n-k}$ for all positive integers $n \geq 2k + 1$. We have $P(2k + 1)$ is true because $F_{2k+1}^{(k)} = (F_2)^{k-1} F_3 = 2^{k-1} \cdot 3 <$

$2^{k-1} \cdot 2^2 = 2^{k+1}$. Assume that $P(\ell)$ is true where ℓ is a positive integer such that $\ell \geq 2k+1$. Then we have $\ell = mk+r$ where $0 \leq r < k$. It follows that $F_\ell^{(k)} = (F_m)^{k-r}(F_{m+1})^r < 2^{\ell-k}$. We consider $\ell + 1 = mk + (r+1)$ and divide this case into two subcases.

Subcase 4.1: $r + 1 < k$. Then we obtain

$$\begin{aligned} F_{\ell+1}^{(k)} &= (F_m)^{k-(r+1)}(F_{m+1})^{r+1} \\ &= \frac{F_m^{k-r}}{F_m} F_{m+1}^r \cdot F_{m+1} \\ &= F_m^{k-r} F_{m+1}^r \cdot \frac{F_{m+1}}{F_m} \\ &< 2^{\ell-k} \cdot 2 = 2^{\ell-k+1}. \end{aligned}$$

Subcase 4.2: $r + 1 = k$. Then $\ell + 1 = (m+1)k$. Thus, we have

$$\begin{aligned} F_{\ell+1}^{(k)} &= F_{(m+1)k}^{(k)} = (F_{m+1})^k \\ &= (F_{m+1})^{k-1}(F_m + F_{m-1}) \\ &= F_m(F_{m+1})^{k-1} + F_{m-1}(F_{m+1})^{k-1} \\ &= F_m(F_{m+1})^{k-1} + \frac{F_{m-1}F_m F_{m+1}^{k-1}}{F_m} \\ &= F_m(F_{m+1})^{k-1}\left(1 + \frac{F_{m-1}}{F_m}\right) \\ &= F_{mk+(k-1)}^{(k)}\left(1 + \frac{F_{m-1}}{F_m}\right) \\ &= F_\ell^{(k)}\left(1 + \frac{F_{m-1}}{F_m}\right) \\ &< 2^{\ell-k} \cdot 2 = 2^{\ell-k+1}. \end{aligned}$$

Hence, $P(\ell+1)$ is true, and so, by the Principle of Mathematical Induction, $P(n)$ is true for all $n > 2k$. □

Theorem 3.2. *Let n, k and ℓ be positive integers where $k \geq 2$ and $\ell \leq k$. The generalized k-Fibonacci numbers satisfy*

$$F_{nk+\ell}^{(k)} = \sum_{i=0}^{\ell} \binom{\ell}{i} F_{nk-i}^{(k)} .$$

Proof: We notice that $F_{nk-\ell}^{(k)} = F_{(n-1)k+(k-i)}^{(k)} = (F_{n-1})^i(F_n)^{k-i}$ for all $i \leq k$. Then we have, for $\ell \leq k$,

$$\begin{aligned} F_{nk+l}^{(k)} &= (F_n)^{k-\ell}(F_{n+1})^\ell \\ &= (F_n)^{k-\ell}(F_n + F_{n-1})^\ell \\ &= (F_n)^{k-\ell} \sum_{i=0}^{\ell} \binom{\ell}{i} F_n^{\ell-i} F_{n-1}^i \\ &= \sum_{i=0}^{\ell} \binom{\ell}{i} F_n^{k-i} F_{n-1}^i \\ &= \sum_{i=0}^{\ell} \binom{\ell}{i} F_{n-1}^i F_n^{k-i} \end{aligned}$$

$$= \sum_{i=0}^{\ell} \binom{\ell}{i} F_{nk-i}^{(k)}.$$

This completes the proof. □

Next, we give some properties for generalized k-Fibonacci numbers.

Corollary 3.1. *Let k and n be positive integers with $k \geq 3$. The generalized k-Fibonacci numbers satisfy*

$$F_{nk+3}^{(k)} = F_{nk+2}^{(k)} + F_{nk-1}^{(k)} + 2F_{nk-2}^{(k)} + F_{nk-3}^{(k)}.$$

Proof: By Theorem 3.2, we obtain

$$F_{nk+3}^{(k)} - F_{nk+2}^{(k)} = \sum_{i=0}^{3} \binom{3}{i} F_{nk-i}^{(k)} - \sum_{i=0}^{2} \binom{2}{i} F_{nk-i}^{(k)} = F_{nk-1}^{(k)} + 2F_{nk-2}^{(k)} + F_{nk-3}^{(k)}.$$ □

We investigate the following theorem by using Cassini's identity [18], that is

$$F_n^2 - F_{n+1}F_{n-1} = (-1)^n \text{ where } F_0 = F_1 = 1.$$

Theorem 3.3. *Let k and n be positive integers with $k \geq 2$. The generalized k-Fibonacci numbers satisfy*

1) $F_{nk}^{(k)} + F_{nk+1}^{(k)} - F_{nk+2}^{(k)} = (-1)^n (F_n)^{k-2}$,

2) $F_{nk-2}^{(k)} + F_{nk-1}^{(k)} - F_{nk}^{(k)} = (-1)^{n+1} (F_n)^{k-2}$.

Proof: 1) Since $F_n^{(k)} = (F_m)^{k-r}(F_{m+1})^r$ for $n = mk + r$, we have $F_{nk+2}^{(k)} = (F_n)^{k-2}(F_{n+1})^2$, $F_{nk+1}^{(k)} = (F_n)^{k-1}F_{n+1}$ and $F_{nk}^{(k)} = (F_n)^k$. Then we obtain

$$\begin{aligned} F_{nk}^{(k)} + F_{nk+1}^{(k)} - F_{nk+2}^{(k)} &= (F_n)^k + (F_n)^{k-1}F_{n+1} - (F_n)^{k-2}(F_{n+1})^2 \\ &= (F_n)^k - (F_n)^{k-2}F_{n+1}(F_{n+1} - F_n) \\ &= (F_n)^k - (F_n)^{k-2}F_{n+1}F_{n-1} \\ &= (F_n)^{k-2}\left((F_n)^2 - F_{n+1}F_{n-1}\right) \\ &= (-1)^n (F_n)^{k-2}. \end{aligned}$$

2) By the same way, we have

$$\begin{aligned} F_{nk-2}^{(k)} + F_{nk-1}^{(k)} - F_{nk}^{(k)} &= F_{(n-1)k+(k-2)}^{(k)} + F_{(n-1)k+(k-1)}^{(k)} - F_{nk}^{(k)} \\ &= (F_{n-1})^2 F_n^{k-2} + F_{n-1}(F_n)^{k-1} - (F_n)^k \\ &= (F_n)^{k-2}\left[(F_{n-1})^2 + F_{n-1}F_n - (F_n)^2\right] \\ &= (F_n)^{k-2}\left[F_{n-1}((F_{n-1}) + F_n) - (F_n)^2\right] \\ &= (F_n)^{k-2}\left[F_{n-1}F_{n+1} - (F_n)^2\right] \\ &= (-1)^{n+1}(F_n)^{k-2}. \end{aligned}$$ □

Corollary 3.2. *Let k and n be positive integers with $k \geq 2$. The generalized k-Fibonacci numbers satisfy*

$$F_{nk+2}^{(k)} = F_{nk+1}^{(k)} + F_{nk-2}^{(k)} + F_{nk-1}^{(k)}.$$

In the next result, we can generalize each term of the k-Lucas sequence.

Theorem 3.4. *Let n and k be positive integers with $k \geq 2$. Then*

$$L_n^{(k)} \begin{cases} = 2^{k-n} & \text{if } n \leq k \\ = 3^r & \text{if } n = mk + r \text{ and } k < n < 2k \\ = 3^k & \text{if } n = 2k \\ < 2 \cdot 3^{n-k-1} & \text{if } n > 2k. \end{cases}$$

Proof: **Case 1**: Let $n \leq k$. We divide this case into two subcases.
Subcase 1.1: $n < k$. Then we have $n = 0 \cdot k + n$. Then we obtain $L_n^{(k)} = (L_0)^{k-n}(L_1)^n = 2^{k-n} \cdot 1 = 2^{k-n}$.
Subcase 1.2: $n = k$. This implies that $L_n^{(k)} = (L_1)^k(L_2)^0 = 1 \cdot 2^0 = 2^{k-k}$.
Case 2: Let $k < n < 2k$. Then $n = k + r$ for all $r \in \{1, 2, \ldots, k-1\}$. This implies that

$$L_n^{(k)} = L_{k+r}^{(k)} = (L_1)^{k-r}(L_2)^r = 1^{k-r} \cdot 3^r = 3^r.$$

Case 3: Let $n = 2k$. Then we obtain $L_{2k}^{(k)} = (L_2)^k(L_3)^0 = 3^k \cdot 1 = 3^k$.
Case 4: Let $P(n)$ be the given statement, that is, $P(n) : L_n^{(k)} < 2 \cdot 3^{n-k-1}$ for all positive integers $n \geq 2k+1$. We have $P(2k+1)$ is true because $L_{2k+1}^{(k)} = (L_2)^{k-1}L_3 = 4 \cdot 3^{k-1} < 6 \cdot 3^{k-1} = 2 \cdot 3^k$. Assume that $P(\ell)$ is true where ℓ is a positive integer such that $\ell \geq 2k+1$. Then we have $\ell = mk + r$ where $0 \leq r < k$. It follows that $L_\ell^{(k)} = (L_m)^{k-r}(L_{m+1})^r < 2 \cdot 3^{\ell-k-1}$. We consider $\ell + 1 = mk + (r+1)$ and divide this case into two subcases.
Subcase 4.1: $r + 1 < k$. Then we obtain

$$\begin{aligned}
L_{\ell+1}^{(k)} &= (L_m)^{k-(r+1)}(L_{m+1})^{r+1} \\
&= \frac{L_m^{k-r}}{L_m} L_{m+1}^r \cdot L_{m+1} \\
&= (L_m)^{k-r}(L_{m+1})^r \cdot \frac{L_{m+1}}{L_m} \\
&< 2 \cdot 3^{\ell-k}.
\end{aligned}$$

Subcase 4.2: $r + 1 = k$. Then $\ell + 1 = (m+1)k$. Thus, we have

$$\begin{aligned}
L_{\ell+1}^{(k)} &= L_{(m+1)k}^{(k)} = (L_{m+1})^k \\
&= (L_{m+1})^{k-1}(L_m + L_{m-1}) \\
&= L_m(L_{m+1})^{k-1} + L_{m-1}(L_{m+1})^{k-1} \\
&= L_m(L_{m+1})^{k-1} + \frac{L_{m-1}L_mL_{m+1}^{k-1}}{L_m} \\
&= L_m(L_{m+1})^{k-1}\left(1 + \frac{L_{m-1}}{L_m}\right) \\
&= L_{mk+(k-1)}^{(k)}\left(1 + \frac{L_{m-1}}{L_m}\right) \\
&= L_\ell^{(k)}\left(1 + \frac{L_{m-1}}{L_m}\right) \\
&< 2 \cdot 3^{\ell-k}.
\end{aligned}$$

Hence, $P(\ell+1)$ is true, and so, by the Principle of Mathematical Induction, $P(n)$ is true for all $n > 2k$. □

In the next results, we are focused on the properties of generalized k-Lucas numbers.

Theorem 3.5. *Let n, k and ℓ be positive integers where $k \geq 2$ and $\ell \leq k$. The generalized k-Lucas numbers satisfy*

$$L_{nk+\ell}^{(k)} = \sum_{i=0}^{\ell} \binom{\ell}{i} L_{nk-i}^{(k)}.$$

Proof: We notice that $L_{nk-\ell}^{(k)} = L_{(n-1)k+(k-i)}^{(k)} = (L_{n-1})^i(L_n)^{k-i}$ for all $i \leq k$. Then we have, for $\ell \leq k$,

$$L_{nk+l}^{(k)} = (L_n)^{k-\ell}(L_{n+1})^\ell$$

$$= (L_n)^{k-\ell}(L_n + L_{n-1})^{\ell}$$
$$= (L_n)^{k-\ell}\sum_{i=0}^{\ell}\binom{\ell}{i}L_n^{\ell-i}L_{n-1}^i$$
$$= \sum_{i=0}^{\ell}\binom{\ell}{i}L_n^{k-i}L_{n-1}^i$$
$$= \sum_{i=0}^{\ell}\binom{\ell}{i}L_{n-1}^i L_n^{k-i}$$
$$= \sum_{i=0}^{\ell}\binom{\ell}{i}L_{nk-i}^{(k)}.$$

This completes the proof. □

We investigate the following theorem by using Cassini's identity [18], that is

$$L_n^2 - L_{n+1}L_{n-1} = (-1)^{n+1}5 \text{ where } L_0 = 2, L_1 = 1.$$

Theorem 3.6. *Let k and n be positive integers with $k \geq 2$. The generalized k-Lucas numbers satisfy*

1) $L_{nk}^{(k)} + L_{nk+1}^{(k)} - L_{nk+2}^{(k)} = (-1)^{n+1}5(L_n)^{k-2}$,

2) $L_{nk-2}^{(k)} + L_{nk-1}^{(k)} - L_{nk}^{(k)} = (-1)^{n}5(L_n)^{k-2}$.

Proof: We can prove in the same way as Theorem 3.3. □

Finally, we present several relationships among Fibonacci, Lucas, generalized k-Fibonacci, and generalized k-Lucas numbers.

Theorem 3.7. *Let n and k be positive integers with $k \geq 2$. Then*

$$(L_n + F_n)^k = \sum_{i=0}^{k}\binom{k}{i}3^{(k-i)}(-1)^i F_{nk-i}^{(k)}.$$

Proof: We have $F_n^{(k)} = (F_m)^{k-r}(F_{m+1})^r$ for $n = mk + r$ $(0 \leq r < k)$. We know that $L_n = F_n + F_{n-2}$. Then we obtain

$$(L_n + F_n)^k = (F_n + F_{n-2} + F_n)^k = (3F_n - F_{n-1})^k$$
$$= \sum_{i=0}^{k}\binom{k}{i}3^{(k-i)}(-1)^i(F_{n-1})^i(F_n)^{k-i}$$
$$= \sum_{i=0}^{k}\binom{k}{i}3^{(k-i)}(-1)^i F_{(n-1)k+(k-i)}^{(k)}$$
$$= \sum_{i=0}^{k}\binom{k}{i}3^{(k-i)}(-1)^i F_{nk-i}^{(k)}.$$ □

Theorem 3.8. *Let n be a positive interger. Then*

$$L_n(F_n)^k = 2F_n^{k+1} - F_nF_{nk-1}^{(k)}.$$

Proof: We have $F_n^{(k)} = (F_m)^{k-r}(F_{m+1})^r$ for $n = mk + r$ $(0 \leq r < k)$. By the identity $L_n = F_n + F_{n-2}$, we obtain

$$L_n(F_n)^k = (F_n + F_{n-2})(F_n)^k$$
$$= F_n^{k+1} + F_{n-2}F_n^k$$
$$= F_n^{k+1} + (F_n - F_{n-1})F_n^k$$

$$
\begin{aligned}
&= F_n^{k+1} + F_n^{k+1} - F_{n-1}F_n^k \\
&= 2F_n^{k+1} - F_n\left(F_{n-1}F_n^{k-1}\right) \\
&= 2F_n^{k+1} - F_nF_{nk-1}^{(k)}. \qquad \square
\end{aligned}
$$

4. **Conclusion.** This paper presents a collection of theorems that demonstrate constraints on the values of the generalized k-Fibonacci and k-Lucas numbers, while also deriving a binomial form representing the sum of Lucas and Fibonacci numbers in the context of the generalized k-Fibonacci sequences. Moreover, we have proven various relationships among the generalized k-Fibonacci numbers, generalized k-Lucas numbers, ordinary Fibonacci and ordinary Lucas numbers where k is a positive integer. In the future, researchers can establish constraints on the values of other generalized Fibonacci and Lucas numbers.

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