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A Study on Essential Fuzzy Ideals of Semirings

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Abstract. In this paper, we define essential fuzzy ideals of semirings and investigate some properties of them. We give some properties of essential ideals and essential fuzzy ideals. Moreover, we show relationships between essential ideals and essential fuzzy ideals.

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1. Introduction

A semiring as the algebraic structure, is definitely a one of generalizations of rings. A semiring was appropriate to ask which properties of rings can be extended to semirings. The concept of semirings was introduced by Vandiver in 1935. One may expect semirings always to be extended to rings, however, Vandiver [1] gave examples of semirings that cannot be embedded in rings. Ideal theory is the main of researching of ring theory and also in semiring theory. A proper ideal of a ring is called essential if it has nonzero intersection with each nonzero ideal. Some properties of essential ideals of rings can see in [2]. Similar to ring theory, an essential ideal of a semiring was similar defined in [3]. The notion of fuzzy sets as the extension of classical sets was introduced by Zadeh [4] in 1965. Fuzzy sets permitted the gradual assessment of the membership of elements and described with the membership valued function of for each element in set to the value in the unit closed interval $[0, 1]$. Fuzzy sets have been applied to many algebraic structures like groups, semigroups, rings, modules and so on. Similar to many algebraic structures, fuzzy semirings were studied in [5]. The concepts of essential ideals were defined and studied in semigroups [6]. Moreover, their various kinds of fuzzifications of essential ideals of semigroups were also defined and studied [6–9]. Fuzzy essential ideals in rings were also studied in [10].

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The purpose of this paper is to define essential fuzzy ideals of semirings. Moreover, we show some relationships between essential ideals and their fuzzifications.

2. Preliminaries

In this section, we will recall in the basic concepts of semirings, fuzzy sets and fuzzy ideals of semirings.

2.1. Semirings

By a *semiring* we shall mean a nonempty set R endowed with two binary operations called the *addition* $+$ and *multiplication* $\cdot$ satisfying the following conditions.

(1) $(R,+)$ is a commutative semigroup.

(2) $(R,\cdot)$ is a semigroup.

(3) The multiplication distributes over the addition both from the left and from the right, that is, $(a+b)c = ac+bc$ and $c(a+b) = ca+cb$ for all $a,b,c \in R$.

A semiring R is called *commutative* if $ab = ba$ for all $a,b \in R$. An element 0 of a semiring R is called a *zero* of S if $0+x = x$ and $x0 = 0x = 0$ for all $x \in R$. A semiring which contains a zero is called a *semiring with zero*. A non-empty subset I of a semiring R is called a *left (resp. right) ideal* of R if for $x,y \in I$ and $r \in R$ imply that $x+y \in I$ and $rx \in I$ (resp. $xr \in I$). If I is both a left and right ideal of R, we say I is a *two-sided ideal*, or simply, an *ideal* of R. For $x \in R$, we let $\langle x\rangle$ be the smallest ideal of R generated by x. We have that $\langle x\rangle = \{nx+ax+xb \mid n \in \mathbb{N}_0 \text{ and } a,b \in R\}$.

Let R be a commutative semiring with zero 0. Let $x \in R\setminus\{0\}$. If $xy = 0$ for some $y \in R\setminus\{0\}$, then x is called a *zero divisor* of R.

2.2. Fuzzy Sets

A fuzzy subset of a set S is a function from S into the closed interval $[0,1]$. Let f and g be any two fuzzy subsets of a set S.

1. The *intersection* of f and g is a fuzzy subset $f \cap g$ of S defined for all $x \in S$ by

$$(f \cap g)(x) = \min\{f(x), g(x)\}.$$

2. The *union* of f and g is a fuzzy subset $f \cup g$ of S defined for all $x \in S$ by

$$(f \cup g)(x) = \max\{f(x), g(x)\}.$$

3. If $f(x) \le g(x)$ for all $x \in S$, we say that f is a *subset* of g and use the notation $f \subseteq g$.

The *support* of a fuzzy subset f of a set S is defined by $supp(f) = \{x \in S \mid f(x) \neq 0\}$. The *characteristic mapping* of a subset A of a set S is a fuzzy subset of S defined by

$$C_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

For any two subsets A and B of S, we have that $C_{A\cap B} = C_A \cap C_B$ and $C_{A\cup B} = C_A \cup C_B$.

A fuzzy set f of a semiring R is called a *fuzzy ideal* of R if for all $x, y \in R$, we have

(1) $f(x + y) \geq \min\{f(x), f(y)\}$,

(2) $f(xy) \geq \max\{f(x), f(y)\}$.

Proposition 1. *A nonempty subset A of a semiring R is an ideal of S if and only if C_A is a fuzzy ideal of R.*

Proposition 2. *Let f be a nonzero fuzzy ideal of a semiring R. Then $supp(f)$ is an ideal of R.*

3. Main Results

Throughout of this section, we let R be a semiring with zero 0. First, we recall the definition of essential ideals of R as follows:

Definition 1. [3] An ideal I of R is called an *essential ideal* of R if $I \cap K \neq \{0\}$ for every nonzero ideal K of R.

Example 1. We consider the semiring $\mathbb{Z}_6$ under the usual addition and multiplication of integers modulo 6. Let $I = \langle\overline{2}\rangle = \{\overline{0}, \overline{2}, \overline{4}\}$ and $J = \langle\overline{3}\rangle = \{\overline{0}, \overline{3}\}$. We see that $I \cap J = \{\overline{0}\}$. Hence I and J are not essential ideals of $\mathbb{Z}_6$.

Proposition 3. *If R is commutative and I is an ideal containing a non zero divisor of R, then I is an essential ideal of R.*

Proof. Suppose that I is not an essential ideal of R. Then there exists a nonzero ideal K of R such that $I \cap K = \{0\}$. Let x is be a non zero divisor of I and $y \in K \setminus \{0\}$. Then $xy \neq 0$ and $xy \in I \cap K$, this is a contradiction. Then I is an essential ideal of R.

Example 2. We have that the semiring $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ under the usual addition and multiplication of integers has no a zero divisor. Hence, by Proposition 3, every nonzero ideal of $\mathbb{N}_0$ is essential.

The following corollary follows from Proposition 3.

Corollary 1. *If R is commutative and I is not an essential ideal of R, then every nonzero element in I is a zero divisor of R.*

However, the converse of Corollary 1 is not generally true.

Example 3. We consider the semiring $\mathbb{Z}_4$ under the usual addition and multiplication of integers modulo 4. Let $I = \langle \overline{2} \rangle = \{\overline{0}, \overline{2}\}$. We see that I is an essential ideal of $\mathbb{Z}_4$ and every nonzero element in I is a zero divisor of R because $\overline{2} \cdot \overline{2} = \overline{0}$.

The two following propositions are some properties of essential ideals of R.

Proposition 4. *Let I be an essential ideal of R. If J is an ideal of R containing I, then J is also an essential ideal of R.*

Proof. Assume that I is an essential ideal of R and J is an ideal of R such that $I \subseteq J$. Let K be any nonzero ideal of R. Thus $I \cap K \neq \{0\}$. Since $I \cap K \subseteq J \cap K$, this implies that $J \cap K \neq \{0\}$. Hence, J is an essential ideal of R.

Proposition 5. *Assume that R is commutative. Let I and J be essential ideals of R. If R has no a zero divisor, then $I \cap J$ is also an essential ideal of R.*

Proof. Since I and J are ideals of R, we have $I \cap J$ is also an ideal of R. Let K be any nonzero ideal of R. Thus $I \cap K \neq \{0\}$. So there exists a nonzero element $x \in I \cap K$. Let $y \in J \setminus \{0\}$. Then $xy \in (I \cap J) \cap K$. By assumption, we have $xy \neq 0$. Thus $(I \cap J) \cap K \neq \{0\}$. Hence, $I \cap J$ is an essential ideal of R.

Definition 2. A fuzzy ideal f of R is called a *nontrivial fuzzy ideal* of R if there exists a nonzero element $x \in R$ such that $f(x) \neq 0$.

By definition of nontrivial fuzzy ideals, the following lemma is obvious.

Lemma 1. *Let g be a fuzzy ideal of R. Then g is a nontrivial fuzzy ideal of R if and only if $supp(g) \neq \{0\}$.*

We define the definition of essential fuzzy ideals of R as follows:

Definition 3. A fuzzy ideal f of R is called an *essential fuzzy ideal* of R if $supp(f \cap g) \neq \{0\}$ for every nontrivial fuzzy ideal g of S.

Example 4. We consider the semiring $\mathbb{N}_0$ under the usual addition and multiplication of integers. Define a fuzzy subset f of $\mathbb{N}_0$ by $f(0) = 1$ and $f(n) = \dfrac{n-1}{n}$ for all $n \in \mathbb{N}$. We have that f is an essential fuzzy ideal of $\mathbb{N}_0$.

Proposition 6. *Let f be an essential fuzzy ideal of R. If h is a fuzzy ideal of R such that $f \subseteq h$, then h is also essential.*

Proof. Assume that f is an essential fuzzy ideal of R and let h be a fuzzy ideal of S such that $f \subseteq h$. Let g be any nontrivial fuzzy ideal of R. Thus $supp(f \cap g) \neq \{0\}$. So $supp(h \cap g) \neq \{0\}$. Hence, h is also an essential fuzzy ideal of R.

The two following theorems show relationships between essential ideals and essential fuzzy ideals of R.

Theorem 1. *A nonzero ideal I of R is essential if and only if C_I is an essential fuzzy ideal of R.*

Proof. Assume that I is an essential ideal of R. By Proposition 1, C_I is a fuzzy ideal of R. We let g be any nontrivial fuzzy ideal of R. By Lemma 1 and Proposition 2, we have $supp(g)$ is a nonzero ideal of S. So $I \cap supp(g) \neq \{0\}$. Then there exists a nonzero element x such that $x \in I \cap supp(g)$, this implies that $(C_I \cap g)(x) \neq 0$. Therefore $x \in supp(C_I \cap g)$. Thus $supp(C_I \cap g) \neq \{0\}$. Hence C_I is an essential fuzzy ideal of R. To prove the converse, we assume that C_I is an essential fuzzy ideal of R. Let K be a nonzero ideal of R. By Lemma 1, we have that C_K is a nontrivial fuzzy ideal of R. Thus $supp(C_I \cap C_K) \neq \{0\}$. Thus $C_{I \cap K} \neq C_{\{0\}}$, this implies that $I \cap K \neq \{0\}$. Hence I is essential.

Theorem 2. *A nontrivial fuzzy ideal f of R is essential if and only if $supp(f)$ is an essential ideal of R.*

Proof. Assume that f is an fuzzy essential ideal of R. Since f is a fuzzy ideal of R, $supp(f)$ is an ideal of R by Proposition 2. Let J be any nonzero ideal of R. By Lemma 1, C_J is a nontrivial fuzzy ideal of R. Since f is essential, $supp(f \cap C_J) \neq \{0\}$. Thus there exists a nonzero element x in R such that $(f \cap C_J)(x) \neq 0$. Thus $f(x) \neq 0$ and $C_J(x) \neq 0$. Hence $x \in supp(f) \cap J$. Then $supp(f) \cap J \neq \{0\}$, this implies that $supp(f)$ is an essential ideal of R. Conversely, assume that $supp(f)$ is an essential ideal of R. Let g be a nontrivial fuzzy ideal of R. Thus $supp(g)$ is a nonzero ideal of R. So $supp(f) \cap supp(g) \neq \{0\}$. Thus there exists a nonzero element x in R such that $x \in supp(f) \cap supp(g)$, so $f(x) \neq 0$ and $g(x) \neq 0$. Therefore, $(f \cap g)(x) \neq 0$. Hence, $supp(f \cap g) \neq \{0\}$. This implies that f is essential.

In the remainder of this section, we will investigate relationships between (minimal, prime, semiprime) essential ideals and (minimal, prime, semiprime) essential fuzzy ideals.

An essential ideal I of R is called *minimal* if for every essential ideal J of R such that $J \subseteq I$, we have $J = I$. An essential fuzzy ideal f of R is called *minimal* if for every essential fuzzy ideal g of R such that $g \subseteq f$, we have $supp(f) = supp(g)$.

Theorem 3. *A nonempty subset S of R is a minimal essential ideal of R if and only if C_S is a minimal essential fuzzy ideal of R.*

Proof. Assume that S is a minimal essential ideal of R. Since S is an essential ideal of R, we have C_S is an essential fuzzy ideal of R by Theorem 1. Let g be any essential fuzzy ideal of R such that $g \subseteq C_S$. Thus $supp(g) \subseteq supp(C_S) = S$. Since g is an essential fuzzy ideal of R, by Theorem 2, $supp(g)$ is an essential ideal of R. Therefore $supp(g) = S$ because S is minimal. This implies that $supp(g) = supp(C_S)$. Hence, C_S is minimal.

To prove the converse, we suppose that C_S is a minimal essential fuzzy ideal of R and let I be an essential fuzzy ideal of R such that $I \subseteq S$. This implies that C_I is an essential fuzzy ideal of R such that $C_I \subseteq C_S$. Since C_S is minimal, $supp(C_I) = supp(C_S)$. Therefore, $I = supp(C_I) = supp(C_S) = S$. Hence, S is minimal.

An essential ideal I of R is called *prime* if $rs \in I$ implies $r \in I$ or $s \in I$ for all $r, s \in R$. An essential fuzzy ideal f of R is called *prime* if $f(rs) \leq \max\{f(r), f(s)\}$ for all $r, s \in R$.

Theorem 4. *A nonempty subset S of R is a prime essential ideal of R if and only if C_S is a prime essential fuzzy ideal of R.*

Proof. Let S be a prime essential ideal of R. By Theorem 1, C_S is an essential fuzzy ideal of R. Let r and s be any two elements of R. If $rs \in S$, then we have that $r \in S$ or $s \in S$ because S is prime. Thus $\max\{C_S(r), C_S(s)\} = 1 \geq C_S(rs)$. Otherwise, if $rs \notin S$, then $C_S(rs) = 0 \leq \max\{C_S(r), C_S(s)\}$. By both two cases, we conclude that C_S is a prime essential fuzzy ideal of R. Conversely, we assume that C_S is a prime essential fuzzy ideal of R. It follows by Theorem 1 that S is an essential ideal of R. Let r and s be two elements of R such that $rs \in S$. This implies that $C_S(rs) = 1$. Since C_S is prime, $C_S(rs) \leq \max\{C_S(r), C_S(s)\}$. Then $\max\{C_S(r), C_S(s)\}$ must be equal to 1 and so $r \in S$ or $s \in S$. Hence, S is a prime essential ideal of R.

An essential ideal I of R is called *semiprime* if for all $r \in R, r^2 \in I$ implies $r \in I$. An essential fuzzy ideal f of R is called *semiprime* if for all $r \in R, f(r^2) \leq f(r)$.

Theorem 5. *A nonempty subset S of R is a semiprime essential ideal of R if and only if C_S is a semiprime essential fuzzy ideal of R.*

Proof. Let S be a semiprime essential ideal of R. By Theorem 1, C_S is an essential fuzzy ideal of R. Let r be any element in R. If $r^2 \in S$, then we have $r \in S$ because S is semiprime. This implies that $C_S(r) = 1$. Hence, $C_S(r) \geq C_S(r^2)$. Otherwise, if $r^2 \notin S$, then $C_S(r^2) = 0 \leq C_S(r)$. By both two cases, we conclude that C_S is a semiprime essential fuzzy ideal of R. To prove the converse, assume that C_S is a semiprime essential fuzzy ideal of R. By Theorem 1, we have that S is an essential ideal of R. Let $r \in R$ such that $r^2 \in S$. So $C_S(r^2) = 1$. Since C_S is semiprime, $C_S(r^2) \leq C_S(r)$. Since $C_S(r^2) = 1$, $C_S(r)$ must be equal 1. Therefore, $r \in S$. Hence, S is a semiprime essential ideal of R.

4. Conclusion

Let R be a semiring with zero. In this paper, we first show some properties of essential ideals of R (Proposition 3-5 and Corollary 1). Next, we introduce the definition of essential fuzzy ideals of R (Definition 3). Finally, we show relationships between essential ideals and essential fuzzy ideals of R in Theorem 1-5.

In the future work, we can define an essential ideal of other algebraic structures and show their properties. Moreover, we will define their fuzzifications of essential ideals and show some relationships between essential ideals and essential fuzzy ideals.

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References

[1] H. S. Vandiver. On some simple types of semi-rings. *Am. Math. Mon.*, 46:22–26, 1939.

[2] D. M. Olson. Upper radicals and essential ideals. *J. Australian Math. Soc.*, 30(4):385–389, 1981.

[3] K. F. Pawar. On essential ideal and radical class. *Int. J. Pure Appl. Math. Sci.*, 5:1–5, 2012.

[4] L. A. Zadeh. Fuzzy sets. *Inform. Control*, 8:338–353, 1965.

[5] J. Ahsan K. Saifullah and M. F.Khan. Fuzzy semirings. *Fuzzy Sets Syst.*, 60(3):302–309, 1993.

[6] S. Baupradist B. Chemat K. Palanivel and R. Chinram. Essential ideals and essential fuzzy ideals in semigroups. *J. Discrete Math. Sci. Crypt.*, 24(1):223–233, 2021.

[7] P. Khamrot A. Iampan and T. Gaketem. Essential intuitionistic fuzzy ideals in semigroups. *IAENG Int. J. Comp. Sci.*, 51(12):2026–2033, 2024.

[8] P. Khamrot and T. Gaketem. Essential bipolar fuzzy ideals in semigroups. *Int. J. Anal. Appl.*, 21:1, 2023.

[9] P. Khamrot and T. Gaketem. Essential interval valued fuzzy ideals in semigroups. *ICIC Express Letters*, 17(9):989–996, 2023.

[10] A. Z. M. Alkatib R. D. Mahmood and M. M. F. Qublan. On fuzzy essential ideals of rings. *Int. Math. Forum*, 8:383–391, 2020.