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On completely regular 2-duo ordered semihypergroups

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Abstract

The concept of duo ordered semihypergroups was introduced by Ardekani and Davvaz in 2018. In this paper, we introduce the concept of n-duo ordered semihypergroups. We show that every duo ordered semihypergroup is an n-duo ordered semihypergroup $(n \geq 2)$, but the converse is not generally true. We investigate the concept of completely regular 2-duo ordered semihypergroups and give their characterizations in terms of (2, 0) -hyperideals, (0, 2) -hyperideals, (2, 2) -hyperideals and (2, 2) -quasi-hyperideals. Finally, we show that in 2-duo ordered semihypergroups, every (2, 2) -hyperideal is quasi-prime.

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1. Introduction

Hyperstructure theory was first introduced in 1934 by Marty [14] and has been studied in the following decades and nowadays by many mathematicians. The beauty of hyperstructure is that in hyperstructures, the composition of two elements is a set. Thus, the notion of algebraic hyperstructures is a generalization of the classical notion of algebraic structures for example, we can see in [2, 9, 11]. The concept of semihypergroups is a generalization of the concept of semigroups. We can see some concepts of semihypergroups in [16]. In [7], Heidari and Davvaz studied a semihypergroup $(S, \circ)$ besides a binary relation $\leq$, where $\leq$ is a partial order relation that satisfies the monotone condition. This structure is called an ordered semihypergroup. Then many authors studied these concepts, for example see [3, 4, 5, 6, 13, 18, 19, 20, 21]. In [8], Kehayopulu introduced the notion of duo ordered semigroups. Recently, Ardekani and Davvaz [1] introduced the notion of duo ordered semihypergroups and discussed some of their properties. In this paper, we introduce the concept of n-duo ordered semihypergroups extending the concept of duo ordered semihypergroups. We also present characterizations of completely regular 2-duo ordered semihypergroups using $(2,0)$-hyperideals, $(0,2)$-hyperideals, $(2,2)$-hyperideals and $(2,2)$-quasi-hyperideals. Finally, we extend the notion of completely regular 2-duo semigroups which was studied studied by Luangchaisri and Changphas in [10] to the completely regular 2-duo ordered semihypergroups.

Let S be a non-empty set and $P^*(S) = P(S) \setminus \{\varnothing\}$ denotes the set of all non-empty subsets of S. The map $\circ : S \times S \to P^*(S)$ is called a *hyperoperation* or a *joint operation* on the set S. A couple $(S, \circ)$ is called a *hypergroupoid*. If $x \in S$ and A, B are two non-empty subsets of S, then we denote

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \quad x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

A hypergroupoid $(S, \circ)$ is called a *semihypergroup* if $x \circ (y \circ z) = (x \circ y) \circ z$ for every $x, y, z \in S$.

Definition 1.1 : [7] Let S be a non-empty set and $\leq$ be an ordered relation on S. The triplet $(S, \circ, \leq)$ is called an ordered semihypergroup if the following conditions are satisfied:

(1) $(S, \circ)$ is a semihypergroup;

(2) $(S, \leq)$ is a partially ordered set; and

(3) for every $x, y, z \in S$, $x \leq y$ implies $x \circ z \leq y \circ z$ and $z \circ x \leq z \circ y$, where $x \circ z \leq y \circ z$ means that for every $a \in x \circ z$ there exists $b \in y \circ z$ such that $a \leq b$.

A non-empty subset A of an ordered semihypergroup S is called a *subsemihypergroup* if $A \circ A \subseteq A$. The set $(A]$ is defined to be the set of all elements t of S such that $t \leq a$ for some a in A, that is $(A] = \{t \in S \mid t \leq a \text{ for some } a \in A\}$. For $A = \{a\}$, we write $(a]$ instead of $(\{a\}]$.

Definition 1.2 : [4] A non-empty subset A of an ordered semihypergroup S is called a left (resp. right) hyperideal of S if

(1) $S \circ A \subseteq A$ (resp. $A \circ S \subseteq A$); and

(2) $(A] = A$, equivalently, for $x \in A$ and $y \in S$, $y \leq x$ implies that $y \in A$.

If A is both a left hyperideal and a right hyperideal of an ordered semihypergroup S, then A is called a *hyperideal (or two-side hyperideal)* of S.

Let S be an ordered semihypergroup and A be a non-empty subset of S. The set $L(A)$ and $R(A)$ are called the left hyperideal and the right hyperideal of S generated by A, respectively. We can easily obtain that

$$L(A) = (A \cup S \circ A] \text{ and } R(A) = (A \cup A \circ S].$$

Definition 1.3 : [12] A non-empty subset Q of an ordered semihypergroup S is called a quasi-hyperideal of S if

(1) $(Q \circ S] \cap (S \circ Q] \subseteq Q$; and

(2) $(Q] = Q$, equivalently, for $x \in Q$ and $y \in S$, $y \leq x$ implies that $y \in Q$.

Definition 1.4 : [13] Let S be an ordered semihypergroup and m, n be any positive integers. Then a subsemihypergroup A of S is called an (m, n)-hyperideal of S if

(1) $A^m \circ S \circ A^n \subseteq A$; and

(2) $(A] = A$, equivalently, for $x \in A$ and $y \in S$, $y \leq x$ implies that $y \in A$.

Definition 1.5 : [12] Let S be an ordered semihypergroup and m, n be any positive integers. Then a subsemihypergroup A of S is called an $(m, 0)$-hyperideal (resp. $(0, n)$-hyperideal) of S if

(1) $A^m \circ S \subseteq A$ (resp. $S \circ A^n \subseteq A$); and

(2) $(A] = A$, equivalently, for $x \in A$ and $y \in S$, $y \leq x$ implies that $y \in A$.

Definition 1.6 : [12] Let S be an ordered semihypergroup and m, n be any positive integers. Then a subsemihypergroup Q of S is called an (m, n)-quasi-hyperideal of S if

(1) $(Q^m \circ S] \cap (S \circ Q^n] \subseteq Q$; and

(2) $(Q] = Q$, equivalently, for $x \in Q$ and $y \in S$, $y \leq x$ implies that $y \in Q$.

Lemma 1.1 : [1, 3, 4, 5, 7] *Let S be an ordered semihypergroup. Then the following statements hold:*

(1) $A \subseteq (A]$ and $((A]] = (A]$ for all $A \subseteq S$;

(2) if $A \subseteq B \subseteq S$, then $(A] \subseteq (B]$;

(3) $(A] \circ (B] \subseteq (A \circ B]$ for all $A, B \subseteq S$;

(4) $(A] \cup (B] = (A \cup B]$ for all $A, B \subseteq S$;

(5) $((A] \circ (B]] = ((A] \circ B] = (A \circ (B]] = (A \circ B]$ for all $A, B \subseteq S$; and

(6) $((A] \circ (B] \circ (C]] = (A \circ B \circ C]$ for all $A, B, C \subseteq S$.

For the sake of simplicity, throughout this paper, we denote $A^n = A \circ A \circ \ldots \circ A$ (n-copies) and $A^0 \circ S = S \circ A^0 = S$.

2. Main Results

An ordered semihypergroup S is called *regular (resp. left regular, right regular)* if for every $a \in S$, $a \in (a \circ S \circ a]$ (resp. $a \in (S \circ a^2]$, $a \in (a^2 \circ S]$).

Definition 2.1 : [15] An ordered semihypergroup S is called completely regular if it is both right regular and left regular.

Lemma 2.1 : [15] *Let S be an ordered semihypergroup. Then the following statements are equivalent:*

(1) S is completely regular;

(2) $A \subseteq (A^2 \circ S \circ A^2]$ for all $A \subseteq S$; and

(3) $a \in (a^2 \circ S \circ a^2]$ for all $a \in S$.

Definition 2.2 : [1] An ordered semihypergroup S is called right (resp. left) duo if the right (resp. left) hyperideals of S are two-sided. S is called duo if it is both right duo and left duo.

Definition 2.3 : Let S be an ordered semihypergroup and let n be a positive integer. Then S is said to be an n-duo ordered semihypergroup if it satisfies the following conditions:

(1) every $(n, 0)$-hyperideal of S is a $(0, n)$-hyperideal of S; and

(2) every $(0, n)$-hyperideal of S is an $(n, 0)$-hyperideal of S.

Lemma 2.2 : *Let S be an ordered semihypergroup and A be a non-empty subset of S. If S is a duo ordered semihypergroup, then the sets $L(A)$ and $R(A)$ coincide.*

Proof: Assume that S is a duo ordered semihypergroup and suppose that $\varnothing \neq A \subseteq S$. Let $x \in L(A) = (A \cup S \circ A] = (A] \cup (S \circ A]$. We have $x \in (A]$ or $x \in (S \circ A]$. If $x \in (A]$, then $x \in (A \cup A \circ S]$. If $x \in (S \circ A]$, then $x \in (S \circ (A \cup A \circ S]]$. Since $(A \cup A \circ S]$ is a right hyperideal of S and S is a duo ordered semihypergroup, $(A \cup A \circ S]$ is a left hyperideal of S, i.e., $S \circ (A \cup A \circ S] \subseteq (A \cup A \circ S]$. So, we obtain

$$x \in (S \circ (A \cup A \circ S]] \subseteq ((A \cup A \circ S]] = (A \cup A \circ S].$$

Thus, $x \in (A \cup A \circ S] = R(A)$. Hence, $L(A) \subseteq R(A)$. The case $R(A) \subseteq L(A)$ can be proved similarly. Therefore, $L(A) = R(A)$.

Theorem 2.3 : *Let S be an ordered semihypergroup. If S is a duo ordered semihypergroup, then S is an n-duo ordered semihypergroup where* $n \geq 2$.

Proof: Assume that S is a duo ordered semihypergroup. Let A be an (n, 0)-hyperideal of S. We will show that A is a (0, n)-hyperideal of S. First, consider

$$\begin{aligned} S \circ A^n &\subseteq (A^n \cup S \circ A^n] = ((A \cup S \circ A) \circ A^{n-1}] \subseteq ((A \cup S \circ A] \circ (A^{n-1}]] \\ &= (L(A) \circ (A^{n-1}]] = (R(A) \circ (A^{n-1}]] = ((A \cup A \circ S] \circ (A^{n-1}]] \\ &= (A^n \cup A \circ S \circ A^{n-1}] \subseteq (A \cup A \circ S \circ A^{n-1}]. \end{aligned}$$

In the same way, we have $S \circ A^{n-1} \subseteq (A \cup A \circ S \circ A^{n-2}]$. Thus,

$$\begin{aligned} S \circ A^n &\subseteq (A \cup A \circ (S \circ A^{n-1})] \subseteq (A \cup (A] \circ (A \cup A \circ S \circ A^{n-2}]] \\ &\subseteq (A \cup A^2 \cup A^2 \circ S \circ A^{n-2}] \subseteq (A \cup A^2 \circ S \circ A^{n-2}]. \end{aligned}$$

Continue in the same manner, we obtain that

$$S \circ A^n \subseteq (A \cup A^n \circ S \circ A^{n-n}] = (A \cup A^n \circ S] \subseteq (A] = A.$$

Thus, $S \circ A^n \subseteq A$. Next, let $a \in A$ and $b \in S$ be such that $b \leq a$. Since $b \leq a$ and $a \in A$ where A is an (n, 0)-hyperideal of S, so we have $b \in A$. Hence, A is a (0, n)-hyperideal of S. Similarly, we can show that every (0, n)-hyperideal of S is an (n, 0)-hyperideal of S. Therefore, S is an n-duo ordered semihypergroup.

The converse of the above theorem is not true in general. We can illustrate it by the following example.

Example 2.4 : Let $S = \{a,b,c,d\}$ with the hyperoperation $\circ$ and the order relation $\leq$ below:

$\circ$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a,b\}$	$\{a\}$
d	$\{a\}$	$\{a\}$	$\{a,b\}$	$\{a\}$

$$\leq := \{(a,a),(a,b),(a,c),(a,d),(b,b),(b,c),(c,c),(d,d)\}.$$

Clearly, $(S,\circ,\leq)$ is an ordered semihypergroup. Moreover, $(S,\circ,\leq)$ is a 2-duo ordered semihypergroup but not a duo ordered semihypergroup because a left hyperideal $\{a,d\}$ of S is not a two-sided hyperideal of S.

Let S be an ordered semihypergroup and A be any non-empty subset of S. Then the (m,n)-hyperideal $[A]_{m,n}$ is called the (m,n)*-hyperideal* of S generated by A. Similarly, $[A]_{m,0}$ and $[A]_{0,n}$ are called the $(m,0)$*-hyperideal* and the $(0,n)$*-hyperideal* of S generated by A, respectively. Thus it is of the form

$$[A]_{m,n} = (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ S \circ A^n].$$

In particular, for $A = \{a\}$, we write $[\{a\}]_{m,n}$ by $[a]_{m,n}$ (see [12, 13]). It is observed that if S is a 2-duo ordered semihypergroup, then $[a]_{0,2} = [a]_{2,0}$ for all $a \in S$.

Theorem 2.5 : *Let S be an ordered semihypergroup. Then S is a completely regular 2-duo ordered semihypergroup if and only if the following conditions hold:*

(1) $((A^2 \cup A^2 \circ S)^2] = A$ *for all (0,2)-hyperideal A of S; and*

(2) $((B^2 \cup S \circ B^2)^2] = B$ *for all (2,0)-hyperideal B of S.*

Proof: Assume that S is a completely regular 2-duo ordered semihypergroup, we will show that the condition (1) holds. Let A be a (0,2)-hyperideal of S. Then $A = (A^2]$ because $A \subseteq (A^2 \circ S \circ A^2] \subseteq (A^2] \subseteq (A] = A$. By the assumption and $A = (A^2]$, we have

$$\begin{aligned} A = (A^2] = ((A \cup A)^2] &\subseteq (((A^2] \cup (A^2 \circ S \circ A^2])^2] \\ &\subseteq ((A^2 \cup A^2 \circ S]^2] = ((A^2 \cup A^2 \circ S)^2] \subseteq (A^2] = A. \end{aligned}$$

Thus, $((A^2 \cup A^2 \circ S)^2] = A.$ The condition (2) can be proved similarly.

Conversely, assume that (1) and (2) hold. First, we will show that S is 2-duo. Let A be a (0,2)-hyperideal of S. We will show that A is a (2,0)-hyperideal of S. Then

$$\begin{aligned} A^2 \circ S &= ((A^2 \cup A^2 \circ S)^2] \circ ((A^2 \cup A^2 \circ S)^2] \circ (S] \subseteq ((A^2 \cup A^2 \circ S)^2] \circ (S] \\ &\subseteq ((A^2 \cup A^2 \circ S) \circ (A^2 \cup A^2 \circ S) \circ S] \subseteq ((A^2 \cup A^2 \circ S) \circ (A^2 \circ S)] \\ &\subseteq ((A^2 \cup A^2 \circ S) \circ (A^2 \cup A^2 \circ S)] = ((A^2 \cup A^2 \circ S)^2] = A. \end{aligned}$$

Thus, $A^2 \circ S \subseteq A$. Clearly, if $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$. Hence, A is a (2,0)-hyperideal of S. Similarly, we can prove that B is a (0,2)-hyperideal of S for all B is a (2,0)-hyperideal of S. Therefore, S is 2-duo.

Next, we will show that S is completely regular. Let $a \in S$. We consider

$$
\begin{aligned}
a \in [a]_{2,0} &= ((([a]_{2,0})^2 \cup ([a]_{2,0})^2 \circ S)^2] \\
&= ((([a]_{2,0})^2 \cup ([a]_{2,0})^2 \circ S) \circ (([a]_{0,2})^2 \cup ([a]_{2,0})^2 \circ S)] \\
&= (((a \cup a^2 \cup a^2 \circ S]^2 \cup (a \cup a^2 \cup a^2 \circ S]^2 \circ S) \circ ((a \cup a^2 \cup S \circ a^2]^2 \cup ([a]_{2,0})^2 \circ S)] \\
&\subseteq (((a^2 \cup a^2 \circ S] \cup (a^2 \cup a^2 \circ S] \circ (S]) \circ ((a^2 \cup S \circ a^2] \cup [a]_{2,0})] \\
&\subseteq (((a^2 \cup a^2 \circ S] \cup (a^2 \circ S]) \circ ((a^2 \cup S \circ a^2] \cup [a]_{0,2})] \\
&= ((a^2 \cup a^2 \circ S] \circ (a \cup a^2 \cup S \circ a^2]] \\
&\subseteq (a^3 \cup a^4 \cup a^2 \circ S \circ a \cup a^2 \circ S \circ a^2] \\
&= (a^3] \cup (a^4] \cup (a^2 \circ S \circ a] \cup (a^2 \circ S \circ a^2].
\end{aligned}
$$

Thus, $a \in (a^3]$ or $a \in (a^4]$ or $a \in (a^2 \circ S \circ a]$ or $a \in (a^2 \circ S \circ a^2]$. If $a \in (a^3]$, then $a \le a^3$. So, we have $a \le a^3 = a^2 \circ a \le a^2 \circ a^3 = a^2 \circ a \circ a^2$. Hence, $a \in (a^2 \circ a \circ a^2] \subseteq (a^2 \circ S \circ a^2]$. Similarly, if $a \in (a^4]$, we obtain $a \in (a^2 \circ S \circ a^2]$. If $a \in (a^2 \circ S \circ a]$, then

$$a \in (a^2 \circ S \circ (a^2 \circ S \circ a]] \subseteq (a^2 \circ S \circ (a^2 \circ S \circ (a^2 \circ S \circ a]]] \subseteq (a^2 \circ S \circ a^2 \circ S \circ a^2 \circ S].$$

Since $(S \circ a^2]$ is a (0,2)-hyperideal of S and S is a 2-duo ordered semihypergroup, then $(S \circ a^2]$ is a (2,0)-hyperideal of S, i.e., $(S \circ a^2]^2 \circ S \subseteq (S \circ a^2]$. So, we obtain

$$a \in (a^2 \circ S \circ a^2 \circ S \circ a^2 \circ S] \subseteq (a^2 \circ (S \circ a^2] \circ (S \circ a^2] \circ S] \subseteq (a^2 \circ (S \circ a^2]] \subseteq (a^2 \circ S \circ a^2].$$

Therefore, in either case, S is completely regular.

Theorem 2.6 : *Let S be an ordered semihypergroup. Then S is a completely regular 2-duo ordered semihypergroup if and only if* $((A^2 \cup A^2 \circ S)^2] = A = ((A^2 \cup S \circ A^2)^2]$ *for any (2,2)-hyperideal A of S.*

Proof: Assume that S is a completely regular 2-duo ordered semihypergroup. Let A be a (2,2)-hyperideal of S. We will show that $((A^2 \cup A^2 \circ S)^2] = A$. Consider

$$
\begin{aligned}
&((A^2 \cup A^2 \circ S)^2] = (A^2 \circ A^2 \cup A^2 \circ A^2 \circ S \cup A^2 \circ S \circ A^2 \cup A^2 \circ S \circ A^2 \circ S] \\
&\subseteq (A \cup A \circ S] \subseteq (A \cup (A^2 \circ S \circ A^2] \circ S] \\
&\subseteq (A \cup (A \circ (A^2 \circ S \circ A^2] \circ S \circ A^2] \circ S] \\
&\subseteq (A \cup A \circ A^2 \circ S \circ A^2 \circ S \circ A^2 \circ S].
\end{aligned}
$$

Since $(S \circ A^2]$ is a (0,2)-hyperideal of S and S is a 2-duo ordered semihypergroup, then $(S \circ A^2]$ is a (2,0)-hyperideal of S, i.e., $(S \circ A^2]^2 \circ S \subseteq (S \circ A^2]$. So, we obtain

$$(A \cup A \circ A^2 \circ S \circ A^2 \circ S \circ A^2 \circ S] \subseteq (A \cup A^3 \circ (S \circ A^2] \circ (S \circ A^2] \circ S]$$
$$\subseteq (A \cup A^3 \circ (S \circ A^2]] \subseteq (A \cup A^2 \circ S \circ A^2] \subseteq (A] = A.$$

Hence, $((A^2 \cup A^2 \circ S)^2] \subseteq A$. By the proof of Theorem 2.5, we have $A = (A^2]$. Then

$$A = (A \circ A] = ((A^2] \circ (A^2]] = (A^4] \subseteq ((A^2 \cup A^2 \circ S)^2].$$

Thus, $((A^2 \cup A^2 \circ S)^2] = A$. In case $A = ((A^2 \cup S \circ A^2)^2]$ can be proved similarly. Therefore, $((A^2 \cup A^2 \circ S)^2] = A = ((A^2 \cup S \circ A^2)^2]$.

Conversely, let A be a (0,2)-hyperideal of S. Then A is a (2,2)-hyperideal of S, since $A^2 \circ S \circ A^2 \subseteq A^2 \circ A \subseteq A^2 \subseteq A$ and $(A] = A$. By assumption, we have $((A^2 \cup A^2 \circ S)^2] = A$. Similarly, let B be a (2,0)-hyperideal of S. Thus, B is a (2,2)-hyperideal of S, and so $((B^2 \cup S \circ B^2)^2] = B$. By Theorem 2.5, we conclude that S is a completely regular 2-duo ordered semihypergroup.

Theorem 2.7 : *Let S be an ordered semihypergroup. Then S is a completely regular 2-duo ordered semihypergroup if and only if $((Q^2 \cup Q^2 \circ S)^2] = Q = ((Q^2 \cup S \circ Q^2)^2]$ for any (2,2)-quasi-hyperideal Q of S.*

Proof: Assume that S is a completely regular 2-duo ordered semihypergroup. Let Q be a (2,2)-quasi-hyperideal of S. We will show that $((Q^2 \cup Q^2 \circ S)^2] = Q$. Then

$$((Q^2 \cup Q^2 \circ S)^2] = (Q^2 \circ Q^2 \cup Q^2 \circ Q^2 \circ S \cup Q^2 \circ S \circ Q^2 \cup Q^2 \circ S \circ Q^2 \circ S] \subseteq (Q^2 \circ S]$$

and

$$((Q^2 \cup Q^2 \circ S)^2] = (Q^2 \circ Q^2 \cup Q^2 \circ Q^2 \circ S \cup Q^2 \circ S \circ Q^2 \cup Q^2 \circ S \circ Q^2 \circ S]$$
$$\subseteq (S \circ Q^2 \circ S \cup Q^2 \circ S \circ Q^2] \subseteq (S \circ Q \circ (Q^2 \circ S \circ Q^2] \circ S \cup S \circ Q^2]$$
$$\subseteq (S \circ Q^2 \circ S \circ Q^2 \circ S \cup S \circ Q^2].$$

Since $(S \circ Q^2]$ is a (0,2)-hyperideal of S and S is a 2-duo ordered semihypergroup, then $(S \circ Q^2]$ is a (2,0)-hyperideal of S. So, we obtain $(S \circ Q^2 \circ S \circ Q^2 \circ S \cup S \circ Q^2] \subseteq ((S \circ Q^2] \circ (S \circ Q^2] \circ S \cup S \circ Q^2] \subseteq ((S \circ Q^2] \cup S \circ Q^2] \subseteq (S \circ Q^2]$. Hence, $((Q^2 \cup Q^2 \circ S)^2] \subseteq (Q^2 \circ S] \cap (S \circ Q^2] \subseteq Q$. By assumption, we have $Q \subseteq (Q^2 \circ S \circ Q^2] \subseteq ((Q^2 \cup Q^2 \circ S)^2]$. Thus, $((Q^2 \cup Q^2 \circ S)^2] = Q$. The case $Q = ((Q^2 \cup S \circ Q^2)^2]$ can be proved similarly. Therefore, $((Q^2 \cup Q^2 \circ S)^2] = Q = ((Q^2 \cup S \circ Q^2)^2]$.

Conversely, let A be a (0,2)-hyperideal of S. Then A is a (2,2)-quasi-hyperideal of S, since $(A^2 \circ S] \cap (S \circ A^2] \subseteq (S \circ A^2] \subseteq A$ and $(A] = A$. By assumption, we have $((A^2 \cup A^2 \circ S)^2] = A$. Similarly, let B be a

(2,0)-hyperideal of S. Thus, B is a (2,2)-quasi-hyperideal of S. Hence, $((B^2 \cup S \circ B^2)^2] = B$. By Theorem 2.5, we conclude that S is a completely regular 2-duo ordered semihypergroup. □

Example 2.8 : Let $S = \{a,b,c,d\}$ with the hyperoperation $\circ$ and the order relation $\leq$ below:

$\circ$	a	b	c	d
a	$\{a,d\}$	$\{a,d\}$	$\{a,d\}$	$\{a\}$
b	$\{a,d\}$	$\{b\}$	$\{a,d\}$	$\{a,d\}$
c	$\{a,d\}$	$\{a,d\}$	$\{c\}$	$\{a,d\}$
d	$\{a\}$	$\{a,d\}$	$\{a,d\}$	$\{d\}$

$$\leq := \{(a,a),(a,b),(a,c),(b,b),(c,c),(d,b),(d,c),(d,d)\}.$$

One can check that $(S,\circ,\leq)$ is an ordered semihypergroup (see [21]). We have $\{a,d\}$, $\{a,b,d\}$, $\{a,c,d\}$ and S are (2,2)-hyperideals of S. Moreover, every (2,2)-hyperideal A of S satisfies the equation $((A^2 \cup A^2 \circ S)^2] = A = ((A^2 \cup S \circ A^2)^2]$. Thus, by Theorem 2.6, S is a completely regular 2-duo ordered semihypergroup.

Lemma 2.9 : *Let S be an ordered semihypergroup. Then S is completely regular if and only if $A = (A^2]$ for any (2,2)-hyperideal A of S.*

Proof: Assume that S is completely regular. Let A be a (2,2)-hyperideal of S. We have

$$\begin{aligned} A &\subseteq (A^2 \circ S \circ A^2] \subseteq (A^2 \circ S \circ (A^2 \circ S \circ A^2] \circ (A^2 \circ S \circ A^2]] \\ &\subseteq (A^2 \circ (S \circ A^2 \circ S) \circ A^2 \circ A^2 \circ S \circ A^2] \subseteq ((A^2 \circ S \circ A^2) \circ (A^2 \circ S \circ A^2)] \\ &\subseteq (A^2] \subseteq (A] = A. \end{aligned}$$

Thus, $A = (A^2]$.

Conversely, assume that $A = (A^2]$ for all (2,2)-hyperideal A of S. Let $a \in S$. Then

$$\begin{aligned} a &\in [a]_{2,2} = (([a]_{2,2})^2] = ((a \cup a^2 \cup a^3 \cup a^4 \cup a^2 \circ S \circ a^2]^2] \\ &\subseteq (a^2 \cup a^3 \cup a^4 \cup a^2 \circ S \circ a^2] = (a^2] \cup (a^3] \cup (a^4] \cup (a^2 \circ S \circ a^2]. \end{aligned}$$

Thus, $a \leq a^2$ or $a \leq a^3$ or $a \leq a^4$ or $a \in (a^2 \circ S \circ a^2]$. Hence, in either case, S is completely regular.

Lemma 2.10 : *Let S be an ordered semihypergroup. If S is completely regular, then* $(A \circ B]$ *is a (2,2)-hyperideal of S for all (2,2)-hyperideals* A, B *of S.*

Proof: Assume that S is completely regular. Let A and B be (2,2)-hyperideals of S. Then

$$\begin{aligned}
&(A \circ B]^2 \circ S \circ (A \circ B]^2 = (A \circ B] \circ (A \circ B] \circ (S] \circ (A \circ B] \circ (A \circ B] \\
&\subseteq (A \circ B \circ S \circ A \circ B] \subseteq (A \circ S \circ A \circ B] \subseteq ((A^2 \circ S \circ A^2] \circ S \circ (A^2 \circ S \circ A^2] \circ B] \\
&\subseteq (A^2 \circ (S \circ A^2 \circ S \circ A^2 \circ S) \circ A^2 \circ B] \subseteq (A^2 \circ S \circ A^2 \circ B] \subseteq (A \circ B].
\end{aligned}$$

Thus, $(A \circ B]^2 \circ S \circ (A \circ B]^2 \subseteq (A \circ B]$ and $((A \circ B]] = (A \circ B]$. Therefore, $(A \circ B]$ is a (2,2)-hyperideal of S.

Definition 2.4 : Let S be an ordered semihypergroup and I be a (2,2)-hyperideal of S. Then I is called quasi-prime if for any (2,2)-hyperideals A, B of S, $A \circ B \subseteq I$ implies that $A \subseteq I$ or $B \subseteq I$; I is called quasi-semiprime if for any (2,2)-hyperideal A of S, $A^2 \subseteq I$ implies that $A \subseteq I$.

Remark 2.11 : It is easy to see that every quasi-prime (2,2)-hyperideal of S is quasi-semiprime (2,2)-hyperideal of S.

Lemma 2.12 : *Let S be an ordered semihypergroup. Then* $A = (A^2]$ *for every (2,2)-hyperideal A of S if and only if any (2,2)-hyperideal of S is quasi-semiprime.*

Proof: Assume that A is a (2,2)-hyperideal of S such that $A = (A^2]$. Let I be a (2,2)-hyperideal of S such that $A^2 \subseteq I$. Then $A = (A^2] \subseteq (I] = I$. Thus, I is a quasi-semiprime (2,2)-hyperideal of S.

Conversely, assume that every (2,2)-hyperideal of S is quasi-semiprime. Let A be a (2,2)-hyperideal of S. Then $(A^2] \subseteq A$. Next, we will show that $A \subseteq (A^2]$. Since

$$(A^2]^2 \circ S \circ (A^2]^2 = (A^2] \circ (A^2] \circ (S] \circ (A^2] \circ (A^2] \subseteq (A^2 \circ S \circ A^2 \circ A] \subseteq (A \circ A] = (A^2]$$

and $((A^2]] = (A^2]$, it follows that $(A^2]$ is a (2,2)-hyperideal of S. By assumption, we have $(A^2]$ is quasi-semiprime. Since $A^2 \subseteq (A^2]$, $A \subseteq (A^2]$. Thus, $A = (A^2]$. □

Theorem 2.13 : *Let S be a 2-duo ordered semihypergroup. Then every (2,2)-hyperideal of S is quasi-prime if and only if S is completely regular and (2,2)-hyperideals of S form a chain by inclusion.*

Proof: Assume that every (2,2)-hyperideal of S is quasi-prime. From Remark 2.11, we know that every quasi-prime (2,2)-hyperideal of S is quasi-semiprime. First, we will show that S is completely regular. By Lemma 2.12, we obtain $A=(A^2]$ for any (2,2)-hyperideal A of S. By Lemma 2.9, we have that S is completely regular. Next, we will show that (2,2)-hyperideals of S form a chain by inclusion. Let A and B be (2,2)-hyperideals of S. By Lemma 2.10, we obtain $(A\circ B]$ is a (2,2)-hyperideal of S. By the assumption, we have that $(A\circ B]$ is quasi-prime. Then there are two cases to be considered:

Case 1: $A\subseteq(A\circ B]$. We have

$$A\subseteq(A\circ B]\subseteq(A\circ(B^2\circ S\circ B^2]]\subseteq(A\circ(B^2\circ S\circ B\circ(B^2\circ S\circ B^2]]]$$
$$\subseteq(S\circ B^2\circ S\circ B^2\circ S\circ B^2].$$

Since $(B^2\circ S]$ is a (2,0)-hyperideal of S and S is a 2-duo ordered semihypergroup, $(B^2\circ S]$ is a (0,2)-hyperideal of S. Thus,

$$(S\circ B^2\circ S\circ B^2\circ S\circ B^2]\subseteq(S\circ(B^2\circ S]\circ(B^2\circ S]\circ B^2]\subseteq((B^2\circ S]\circ B^2]$$
$$\subseteq(B^2\circ S\circ B^2]\subseteq(B]=B.$$

Hence, $A\subseteq B$.

Case 2: $B\subseteq(A\circ B]$. Then

$$B\subseteq(A\circ B]\subseteq((A^2\circ S\circ A^2]\circ B]\subseteq(((A^2\circ S\circ A^2]\circ A\circ S\circ A^2]\circ B]$$
$$\subseteq(A^2\circ S\circ A^2\circ S\circ A^2\circ S].$$

Since $(S\circ A^2]$ is a (0,2)-hyperideal of S and S is a 2-duo ordered semihypergroup, then $(S\circ A^2]$ is a (2,0)-hyperideal of S. Thus,

$$(A^2\circ S\circ A^2\circ S\circ A^2\circ S]\subseteq(A^2\circ(S\circ A^2]\circ(S\circ A^2]\circ S]\subseteq(A^2\circ(S\circ A^2]]$$
$$\subseteq(A^2\circ S\circ A^2]\subseteq(A]=A.$$

Hence, $B\subseteq A$. From both cases, we conclude that (2,2)-hyperideals of S form a chain by inclusion.

Conversely, assume that S is completely regular and (2,2)-hyperideals of S form a chain by inclusion. We will show that every (2,2)-hyperideal of S is quasi-prime. Let A,B,I be (2,2)-hyperideals of S such that $A\circ B\subseteq I$. By Lemma2.9, $A=(A^2]$ and $B=(B^2]$. If $A\subseteq B$, then $A=(A^2]\subseteq(A\circ B]\subseteq(I]=I$. Similarly, if $B\subseteq A$, we have $B=(B^2]\subseteq(A\circ B]\subseteq(I]=I$. Thus, I is quasi-prime. □

Open Problems. We can generalize the results of this paper to the results in po-ternary semihypergroups (see [17]).

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