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On left and right bases of po-Gamma-semigroups

Wichayaporn Jantanan †
Nipawan Trihera §
Natee Raikham ‡
Department of Mathematics
Faculty of Science
Buriram Rajabhat University
Buriram 31000
Thailand

Ronnason Chinram @
Algebra and Application Research Unit
Division of Computational Science
Faculty of Science
Prince of Songkla University
Hat Yai, Songkhla 90110
Thailand

Pattarawan Petchkaew *
Mathematics Program
Faculty of Science and Technology
Songkhla Rajabhat University
Songkhla 90000
Thailand

Abstract

The concepts of left and right bases of Γ-semigroups were introduced by Changphas and Kummoon. In this paper, we introduce the concepts of the left and right bases of po-Γ-semigroups and extend the results in Γ-semigroups to po-Γ-semigroups. We focus on the structure of po-Γ-semigroups containing the right bases. In addition, we prove that all

† *E-mail:* wichayaporn.jan@bru.ac.th
§ *E-mail:* nipawanbm15@gmail.com
‡ *E-mail:* nateeraikham04@gmail.com
@ *E-mail:* ronnason.c@psu.ac.th
* *E-mail:* pattarawan.pe@gmail.com (Corresponding Author)

of the right bases of a po-Γ-semigroup have the same cardinality and conclude that a non-empty po-Γ-semigroup eliminating the union of its all right bases is a left Γ-ideal of such po-Γ-semigroup.

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1. Introduction

The concept of semigroups is the one of the algebraic structures which was widely studied. There are many generalizations, for example, Γ-semigroup, ordered semigroups, ternary semigroups, etc. Furthermore, semigroup theories were also applied in decision making [7]. In 1955, Tamura [9] introduced the notion of a right (left) base of semigroups as follows:

Definition 1.1 : A non-empty subset A of a semigroup S is called a right (left) base of S if it satisfies the following two conditions: $S = A \cup SA (S = A \cup AS)$ and if B is a subset of A such that $S = B \cup SB (S = B \cup BS)$ then $B = A$.

Later, Fabrici [2] examined the structure of semigroups containing the right bases. Recently, Changphas and Kummoon [1] introduced the notion of the left and right bases of Γ-semigroups.

In this paper, we introduce the concepts of left and right bases of po-Γ-semigroups and extend the results in Γ-semigroups to po-Γ-semigroups. First of all, we provide some definitions notations and results which will be used for this paper.

Sen [5] defined Γ-semigroups as a generalization of semigroups as follows:

Definition 1.2 : Let S and Γ be any two non-empty sets. Then S is called a Γ-semigroup if there is a mapping from $S \times \Gamma \times S$ into S, written as (a, α, b) by $a\alpha b$, such that $(a\gamma b)\beta c = a\gamma(b\beta c)$ for all $a, b, c \in S$ and all $\alpha, \beta \in \Gamma$.

Sen and Seth [6] introduced the concept of po-Γ-semigroups as follows:

Definition 1.3 : A Γ-semigroup S is called a po-Γ-semigroup (ordered Γ-semigroup) if there is a relation $\leq$ on S such that $x \leq y$ implies $x\gamma z \leq y\gamma z$ and $z\gamma x \leq z\gamma y$ for all $x, y, z \in S$ and $\gamma \in \Gamma$.

Throughout this paper, S stands for a po-Γ-semigroup. For non-empty subsets A and B of S, the set product $A\Gamma B$ of A and B, and the subset $(A]$ of S are defined by :

$$A\Gamma B = \{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\},$$

and

$$(A] = \{x \in S \mid x \leq y \text{ for some } y \in A\}.$$

For $a \in S$, we write $a\Gamma B$, $B\Gamma a$ instead of $\{a\}\Gamma B$, $B\Gamma\{a\}$ and for $A = \{x\}$, we write $(x]$ instead of $(\{x\}]$.

Definition 1.4 : A non-empty subset A of S is called a Γ-subsemigroup of S if $A\Gamma A \subseteq A$ or $a\gamma b \in A$ for all $a, b \in A$ and $\gamma \in \Gamma$.

Definition 1.5 : A non-empty subset A of S is called a left (resp. right) Γ-ideal of S if $S\Gamma A \subseteq A$ (resp. $A\Gamma S \subseteq A$) and if $a \in A$ and $b \in S$ such that $b \leq a$ then $b \in A$.

Definition 1.6 : A left Γ-ideal A of S is called proper if $A \neq S$. A proper left Γ-ideal B of S is called maximal if for any left Γ-ideal A of S such that $B \subseteq A$ implies $B = A$ or $A = S$.

Lemma 1.1 : [4] *Let S be a po-Γ-semigroup and $\{A_i \mid i \in I\}$ a non-empty family of left Γ-ideals of S, then the following statements hold :*

(1) If $\cap \{A_i \mid i \in I\} \neq \emptyset$, then the set $\cap \{A_i \mid i \in I\}$ is a left Γ-ideal of S.

(2) The set $\cup \{A_i \mid i \in I\}$ is a left Γ-ideal of S.

The intersection of all left Γ-ideals of a po-Γ-semigroup S, if it is non-empty, is a left Γ-ideal of S. Thus, for a non-empty subset A of S, the intersection of all left Γ-ideals of S containing A, denoted by $L(A)$, is a left Γ-ideal of S containing A, and it is of the form

$$L(A) = (A \cup S\Gamma A].$$

In particular, for an element $a \in S$, we write $L(a)$ instead of $L(\{a\})$ which is called the *principal left Γ-ideal* of S generated by a. Thus

$$L(a) = (a \cup S\Gamma a].$$

Lemma 1.2 : *Let S be a po-Γ-semigroup. Then the following statements hold:*

(1) $A \subseteq (A]$ for any $A \subseteq S$.

(2) *If* $A \subseteq B \subseteq S$, *then* $(A] \subseteq (B]$.

(3) $(A]\Gamma(B] \subseteq (A\Gamma B]$ *for all subsets A and B of S.*

(4) $((A]] = (A]$ *for all* $A \subseteq S$.

(5) *For every left* Γ*-ideal T of S,* $(T] = T$.

(6) $(S\Gamma a]$ *is a left* Γ*-ideal of S for every* $a \in S$.

(7) $(A \cup B] = (A] \cup (B]$ *for all subsets A and B of S.*

(8) *If A and B are left* Γ*-ideals of S, then* $A \cup B$ *is a left* Γ*-ideal of S.*

2. Main Results

We begin this section with the definition of a right base of a po-Γ-semigroup as follows:

Definition 2.1 : Let *S* be a po-Γ-semigroup. A non-empty subset *A* of *S* is called a right base of *S* if it satisfies the following two conditions :

(i) $S = (A \cup S\Gamma A]$, i.e. $S = L(A)$;

(ii) if *B* is a subset of *A* such that $S = L(B)$, then $B = A$.

For a left base of *S* defined dually.

Example 2.1 : Let $S = \{a, b, c, d\}$ and $\Gamma = \{\gamma\}$ with the multiplication defined by

γ	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	c	c	c
d	a	d	d	d

and $\leq = \{(a,a),(b,b),(c,c),(d,d),(b,a),(c,a),(d,a)\}$.

Then *S* is a po-Γ-semigroup [8]. The right bases of *S* are $A = \{a\}$, $B = \{b\}$, $C = \{c\}$ and $D = \{d\}$. The left bases of *S* are the same as the right bases of *S*.

Example 2.2 : Let $S = \{a, b, c, d\}$ and $\Gamma = \{\gamma\}$ with the multiplication defined by

γ	a	b	c	d
a	b	b	d	d
b	b	b	d	d
c	d	d	c	d
d	d	d	d	d

and $\leq = \{(a,a),(b,b),(c,c),(d,d),(a,b),(d,b),(d,c)\}$.

Then S is a po-Γ-semigroup [8]. The right bases of S are $A = \{a,c\}$ and $B = \{b,c\}$. The left bases of S are the same as the right bases of S.

Lemma 2.3 : *Let A be a right base of a po-Γ-semigroup S and let $a,b \in A$. If $a \in (S\Gamma b]$, then $a = b$.*

Proof : Assume that $a,b \in A$ such that $a \in (S\Gamma b]$, and suppose that $a \neq b$. Let $B = A \setminus \{a\}$, then $B \subset A$. Since $a \neq b$, we obtain $b \in B$. We will show that $L(A) \subseteq L(B)$. Let $x \in L(A) = (A \cup S\Gamma A]$. Then $x \leq z$ for some $z \in A \cup S\Gamma A$. Let $z \in A$. If $z \neq a$, then $z \in B \subseteq (B \cup S\Gamma A]$. Since $x \leq z$ and $z \in (B \cup S\Gamma B]$, then $x \in ((B \cup S\Gamma B]] = (B \cup S\Gamma B]$. So $x \in L(B)$. If $z = a$, then by assumption, we have $z = a \in (S\Gamma B] \subseteq (B \cup S\Gamma B]$. Since $x \leq z$ and $z \in (B \cup S\Gamma B]$, then $x \in ((B \cup S\Gamma B]] = (B \cup S\Gamma B]$. So $x \in L(B)$. Next, let $z \in S\Gamma A$, then $z = s\gamma c$ for some $s \in S$, $\gamma \in \Gamma$ and $c \in A$. If $c = a$, then $z = s\gamma a \in S\Gamma(S\Gamma b] = (S]\Gamma(S\Gamma b] \subseteq (S\Gamma(S\Gamma b)] \subseteq (S\Gamma b] \subseteq (B \cup S\Gamma B]$. Since $x \leq z$ and $z \in (B \cup S\Gamma B]$. We obtain $x \in ((B \cup S\Gamma B]] = (B \cup S\Gamma B]$. So $x \in L(B)$. If $c \neq a$, then $z = s\gamma c \in S\Gamma B \subseteq (B \cup S\Gamma B]$. Since $x \leq z$ and $z \in (B \cup S\Gamma B]$, thus $x \in ((B \cup S\Gamma B]] = (B \cup S\Gamma B]$. So $x \in L(B)$. Hence $L(A) \subseteq L(B)$. By $S = L(A) \subseteq L(B) \subseteq S$, it follows that $L(B) = S$. This is a contradiction. Therefore, $a = b$. □

Definition 2.2 : Let S be a po-Γ-semigroup. Define a **quasi-order** on S by, for any $a,b \in S$,

$$a \leq_L b \Leftrightarrow L(a) \subseteq L(b).$$

The symbol $a <_L b$ stands for $a \leq_L b$ but $a \neq b$.

The following example shows that $\leq_L$ is not a partial order in general.

Example 2.4 : By Example 2.1, we have $L(b) \subseteq L(c)$ i.e., $b \leq_L c$ and $L(c) \subseteq L(b)$ i.e., $c \leq_L b$ but $b \neq c$. Hence $\leq_L$ is not a partial order.

Lemma 2.5 : *Let S be a po-Γ-semigroup. For any $a,b \in S$, if $a \leq b$, then $a \leq_L b$.*

Proof : Let $a,b\in S$ such that $a\leq b$. We will show that $a\leq_L b$, i.e., $L(a)\subseteq L(b)$. Suppose that $x\in L(a)$. Since $x\in(a\cup S\Gamma a]$, then $x\leq y$ for some $y\in a\cup S\Gamma a$. We have $y=a$ or $y\in S\Gamma a$. If $y=a$, then $x\leq a\leq b$. We obtain $x\leq b$ for some $b\in(b\cup S\Gamma b]$. So $x\in((b\cup S\Gamma b]]=(b\cup S\Gamma b]$. Thus $x\in L(b)$. If $y\in S\Gamma a$, then $y=s\gamma a$ for some $s\in S$, $\gamma\in\Gamma$. Since $a\leq b$, then $s\gamma a\leq s\gamma b$ and $s\gamma b\in S\Gamma b\subseteq(b\cup S\Gamma b]$. Hence $y=s\gamma a\in((b\cup S\Gamma b]]=(b\cup S\Gamma b]$. Since $x\leq y$ and $y\in(b\cup S\Gamma b]$, we get $x\in((b\cup S\Gamma b]]=(b\cup S\Gamma b]$. Thus $x\in L(b)$. Therefore, $L(a)\subseteq L(b)$, i.e., $a\leq_L b$. □

Nevertheless, the converse of Lemma 2.5 is not true in general. By Example 2.1, we have $L(a)\subseteq L(b)$ i.e., $a\leq_L b$ but $a\not\leq b$.

Theorem 2.6 : *A non-empty subset A of a po-* Γ *-semigroup S is a right base of S if and only if A satisfies the following two conditions :*

(1) for any $x\in S$ *there exists* $a\in A$ *such that* $x\leq_L a$;

(2) for any two distinct elements $a,b\in A$ *neither* $a\leq_L b$ *nor* $b\leq_L a$.

Proof : Assume that A is a right base of S. Then $S=L(A)$. Let $x\in S$, then $x\in(A\cup S\Gamma A]$. Since $x\in(A\cup S\Gamma A]$, we have $x\leq y$ for some $y\in A\cup S\Gamma A$. Thus $y\in A$ or $y\in S\Gamma A$. If $y\in A$ and $x\leq y$, then $x\leq_L y$ by Lemma 2.5. If $y\in S\Gamma A$, then $y=s\gamma a$ for some $s\in S$, $\gamma\in\Gamma$ and $a\in A$. Since $y\in S\Gamma a\subseteq(a\cup S\Gamma a]$ and $S\Gamma y\subseteq S\Gamma(S\Gamma a)=(S\Gamma S)\Gamma a\subseteq S\Gamma a\subseteq(a\cup S\Gamma a]$, we obtain $y\cup S\Gamma y\subseteq(a\cup S\Gamma a]$. So, $(y\cup S\Gamma y]\subseteq((a\cup S\Gamma a]]=(a\cup S\Gamma a]$. Thus $L(y)\subseteq L(a)$, i.e., $y\leq_L a$. Since $x\leq y$, we have $x\leq_L y$. So $x\leq_L y\leq_L a$ by Lemma 2.5. Thus $x\leq_L a$. Hence the condition (1) holds. Let $a,b\in A$ such that $a\neq b$. Suppose $a\leq_L b$. We set $B=A\setminus\{a\}$. Then $b\in B$. Let $x\in S$, by (1), there exists $c\in A$ such that $x\leq_L c$. Since $c\in A$, there are two cases to consider. If $c=a$, then $x\leq_L b$. So $x\in L(x)\subseteq L(b)\subseteq L(B)$. Thus $S=L(B)$. This is a contradiction. If $c\neq a$, then $c\in B$. Hence $x\in L(x)\subseteq L(c)\subseteq L(B)$. So $S=L(B)$. This is a contradiction. The case $b\leq_L a$ can be proved similarly. Hence the condition (2) holds.

Conversely, assume that the conditions (1) and (2) hold. We will show that $S=L(A)$. Let $x\in S$, by (1), there exists $a\in A$ such that $x\leq_L a$, i.e. $L(x)\subseteq L(a)$. Then $x\in L(x)\subseteq L(a)\subseteq L(A)$. Thus $S\subseteq L(A)$, and $S=L(A)$. Next, we will show that A is a minimal subset of S with the property $S=L(A)$. Let $B\subset A$ such that $S=L(B)$. Then there exists $a\in A$ and $a\notin B$. Since $a\in A\subseteq S=(B\cup S\Gamma B]=(B]\cup(S\Gamma B]$. If $a\in(B]$, then $a\leq y$ for some $y\in B$, by Lemma 2.5, we obtain $a\leq_L y$. This is a contradiction. Thus $a\notin(B]$, and $a\in(S\Gamma B]$. Since $a\in(S\Gamma B]$, we have $a\leq c$ for some $c\in S\Gamma B$. Let $c=s\gamma b$ for some $s\in S$, $\gamma\in\Gamma$ and $b\in B$. Since $a\leq c$ and $c=s\gamma b\in S\Gamma b\subseteq b\cup S\Gamma b$.

Then $a \in (b \cup S\Gamma b]$, so $S\Gamma a \subseteq S\Gamma(b \cup S\Gamma b] = (S]\Gamma(b \cup S\Gamma b] \subseteq (S\Gamma(b \cup S\Gamma b)] = (S\Gamma b \cup S\Gamma(S\Gamma b)] = (S\Gamma b \cup (S\Gamma S)\Gamma b] \subseteq (S\Gamma b \cup S\Gamma b] \subseteq (b \cup S\Gamma b]$. Thus $a \cup S\Gamma a \subseteq (b \cup S\Gamma b]$, and $(a \cup S\Gamma a] \subseteq ((b \cup S\Gamma b]] = (b \cup S\Gamma b]$. Hence $L(a) \subseteq L(b)$, i.e., $a \leq_L b$. This is a contradiction. Therefore, A is a right base of S.

If a right base A of a po-Γ-semigroup S is a left Γ-ideal of S, then

$$S = (A \cup S\Gamma A] = (A \cup A] = (A] = A.$$

Hence $S = A$. The converse statement is obvious. Then we can conclude that :

Theorem 2.7 : *A right base A of a po-Γ-semigroup S is a left Γ-ideal of S if and only if* $A = S$.

Definition 2.3 : A po-Γ-semigroup S is said to be right singular if $x\gamma y = y$ for all $x, y \in S$ and $\gamma \in \Gamma$.

In Example 2.2, it is observed that a right base of a po-Γ-semigroup is not necessary to be a Γ-subsemigroup of such po-Γ-semigroup. This leads to get the result of Theorem 2.8.

Theorem 2.8 : *A right base A of a po-Γ-semigroup S is a Γ-subsemigroup of S if and only if A is right singular.*

Proof : Assume that A is a Γ-subsemigroup of S. Let $a, b \in A$, and let $\gamma \in \Gamma$. By assumption, $a\gamma b \in A$. Setting $a\gamma b = c$ for some $c \in A$. Since $c = a\gamma b \in S\Gamma b \subseteq (S\Gamma b]$, by Lemma 2.3, we have $c = b$. Thus $a\gamma b = b$. Therefore A is right singular. The converse statement is clear. □

Let S be a po-Γ-semigroup and let $\alpha \in \Gamma$. An element e of S is called α-idempotent of S if $e\alpha e = e$. Let $E_\alpha(S)$ denoted the set of all α-idempotents of S. Let $E(S) := \bigcup_{\alpha\in\Gamma} E_\alpha(S)$. By Theorem 2.8, we obtain the following corollary.

Corollary 2.9 : *If a right base A of a po-Γ-semigroup S is a Γ-subsemigroup of S, then* $E(S) \neq \varnothing$.

In Example 2.1 and Example 2.2, it is observed that the cardinality of right bases are the same. However, it turns out that this is true in general, and we will prove in Theorem 2.10.

Theorem 2.10 : *The right bases of a po-Γ-semigroup S have the same cardinality.*

Proof : Let A and B be right bases of a po-Γ-semigroup S. Let $a \in A$. Since B is a right base of S, by Theorem 2.6(1), then there exists $b \in B$ such that $a \leq_L b$. Similarly, since A is a right base of S, there exists $a' \in A$ such that $b \leq_L a'$. So $a \leq_L b \leq_L a'$, and $a \leq_L a'$. By Theorem 2.6(2), $a = a'$. Hence $L(a) = L(b)$. Now, define a mapping

$$\varphi : A \rightarrow B; \varphi(a) = b$$

for all $a \in A$. First, we will show that φ is well-defined, let $a_1, a_2 \in A$ such that $a_1 = a_2$, $\varphi(a_1) = b_1$, and $\varphi(a_2) = b_2$, for some $b_1, b_2 \in B$. Then $L(a_1) = L(b_1)$ and $L(a_2) = L(b_2)$. Since $a_1 = a_2$, then $L(a_1) = L(a_2)$. Hence $L(a_1) = L(a_2) = L(b_1) = L(b_2)$. We have $b_1 \leq_L b_2$ and $b_2 \leq_L b_1$, and so $b_1 = b_2$ by Theorem 2.6(2). Thus $\varphi(a_1) = \varphi(a_2)$. Therefore, φ is well-defined. Next, we will show that φ is one-to-one. Let $a_1, a_2 \in A$ such that $\varphi(a_1) = \varphi(a_2)$. Then $\varphi(a_1) = \varphi(a_2) = b$ for some $b \in B$. We obtain $L(a_1) = L(a_2) = L(b)$. Since $L(a_1) = L(a_2)$, so $a_1 \leq_L a_2$ and $a_2 \leq_L a_1$. Hence $a_1 = a_2$. Therefore, φ is one-to-one. Finally, we will show that φ is onto. Let $b \in B$, then there exists $a \in A$ such that $b \leq_L a$. Similarly, there exists $b' \in B$ such that $a \leq_L b'$. Then $b \leq_L a \leq_L b'$, i.e., $b \leq_L b'$. By Theorem 2.6(2), we get $b = b'$. So $L(b) \subseteq L(a)$ and $L(a) \subseteq L(b)$. Thus $L(a) = L(b)$. Therefore, φ is onto. This completes the proof. □

Theorem 2.11 : *Let A be a right base of a po-Γ-semigroup of S, and let $a \in A$. If $L(a) = L(b)$ for some $b \in S$ such that $a \neq b$, then b belongs to some right base of S which is different from A.*

Proof : Assume that $L(a) = L(b)$ for some $b \in S$ such that $a \neq b$. Setting $B = (A \setminus \{a\}) \cup \{b\}$, then $B \neq A$. We will show that B is a right base of S by using Theorem 2.6. Now, let $x \in S$. Since A is a right base of S, by Theorem 2.6(1), $x \leq_L c$ for some $c \in A$. If $c \neq a$, then $c \in B$. If $c = a$, then $L(c) = L(a)$. Since $L(a) = L(b)$, we have $L(c) = L(b)$, i.e., $c \leq_L b$. So $x \leq_L c \leq_L b$. Thus $x \leq_L b$ where $b \in B$. Next, let $b_1, b_2 \in B$ such that $b_1 \neq b_2$. There are four cases to consider :

Case 1 : $b_1 \neq b$ and $b_2 \neq b$. Then $b_1, b_2 \in A$. Since A is a right base of S, neither $b_1 \leq_L b_2$ nor $b_2 \leq_L b_1$.

Case 2 : $b_1 \neq b$ and $b_2 = b$. Then $L(b_2) = L(b)$. If $b_1 \leq_L b_2$, then $L(b_1) \subseteq L(b_2) = L(b) = L(a)$. Thus $b_1 \leq_L a$ where $b_1, a \in A$. This is a contradiction. If $b_2 \leq_L b_1$, then $L(a) = L(b) = L(b_2) \subseteq L(b_1)$. Thus $a \leq_L b_1$ where $b_1, a \in A$. This is a contradiction.

Case 3 : $b_1 = b$ and $b_2 \neq b$. Then $L(b_1) = L(b)$. If $b_1 \leq_L b_2$, then $L(a) = L(b) = L(b_1) \subseteq L(b_2)$. Thus $a \leq_L b_2$ where $b_2, a \in A$. This is a contradiction. If $b_2 \leq_L b_1$, then $L(b_2) \subseteq L(b_1) = L(b) = L(a)$. Thus $b_2 \leq_L a$ where $b_2, a \in A$. This is a contradiction.

Case 4 : $b_1 = b$ and $b_2 = b$. This is impossible.

Therefore, we can conclude that B is a right base of S which $B \neq A$. □

Theorem 2.12 : *Let C be the union of all right bases of a po-Γ-semigroup S. If $S \setminus C$ is non-empty, then $S \setminus C$ is a left Γ-ideal of S.*

Proof : Assume that $S \setminus C \neq \emptyset$. We will show that $S \setminus C$ is a left Γ-ideal of S. First, let $x \in S$, $\gamma \in \Gamma$ and $a \in S \setminus C$. To show that $x\gamma a \in S \setminus C$. Suppose that $x\gamma a \notin S \setminus C$. Then $x\gamma a \in C$. Thus $x\gamma a \in A$ for some a right base A of S. Let $x\gamma a = b$ for some $b \in A$. Then $b = x\gamma a \in S\Gamma a \subseteq (a \cup S\Gamma a]$, $S\Gamma b \subseteq S\Gamma(S\Gamma a) = (S\Gamma S)\Gamma a \subseteq S\Gamma a \subseteq (a \cup S\Gamma a]$. So $b \cup S\Gamma b \subseteq (a \cup S\Gamma a]$, and $(b \cup S\Gamma b] \subseteq (a \cup S\Gamma a]$. Thus $L(b) \subseteq L(a)$. If $L(b) = L(a)$, by Theorem 2.11, we have $a \in C$. This is a contradiction. Hence $L(b) \subset L(a)$, i.e., $b <_L a$. Since A is a right base of S, there exists $b' \in A$ such that $a \leq b'$. We have $b <_L a \leq_L b'$, and $b \leq_L b'$ where $b, b' \in A$. This is a contradiction. Thus $x\gamma a \in S \setminus C$. Next, let $y \in S \setminus C$ and $z \in S$ such that $z \leq y$. We will show that $z \in S \setminus C$. If $z \in C$, then $z \in B$ for some a right base B of S. Let $c \in B$ such that $y \leq_L c$. Since $z \leq y$, by Lemma 2.5, we have $z \leq_L y$. So $z \leq_L c$ where $z, c \in B$. This is a contradiction. Thus $z \notin C$, i.e., $z \in S \setminus C$. Therefore $S \setminus C$ is a left Γ-ideal of S. □

In Example 2.2, the right bases of S are $A = \{a, c\}$ and $B = \{b, c\}$. We have set S eliminating the union of all right bases of S denoted by $S \setminus C$ and $S \setminus C = \{d\}$ is a left Γ-ideal of S, but it is not a maximal proper left Γ-ideal of S. In the following theorem we shall find the conditions for leading $S \setminus C$ is a maximal proper left Γ-ideal of S.

Theorem 2.13 : *Let C be the union of all right bases of a po-Γ-semigroup S. Then $S \setminus C$ is a maximal proper left Γ-ideal of S if and only if $C \neq S$ and $C \subseteq L(a)$ for all $a \in C$.*

Proof : Let $S \setminus C$ be a maximal proper left Γ-ideal of S. Then $C \neq S$. Let $a \in C$. Suppose $C \nsubseteq L(a)$. Then there exists $b \in C$ such that $b \notin L(a)$. Since $b \notin S \setminus C$, and $b \in S$, then $(S \setminus C) \cup L(a) \subset S$. Thus $(S \setminus C) \cup L(a)$ is a proper left Γ-ideal of S. Hence $S \setminus C \subseteq (S \setminus C) \cup L(a)$. This contradicts to the maximality of $S \setminus C$. Therefore, $C \subseteq L(a)$ for all $a \in C$.

Conversely, let $C \neq S$ and $C \subseteq L(a)$ for all $a \in C$. We obtain $C \subset S$, $S \setminus C \subset S$. Since $S \setminus C \neq \varnothing$, by Theorem 2.12, $S \setminus C$ is a proper left Γ-ideal of S. Let A be a left Γ-ideal of S such that $S \setminus C \subseteq A \subseteq S$. Suppose that $S \setminus C \neq A$. Since $S \setminus C \subset A$, then there exists $x \in A$ such that $x \notin S \setminus C$, i.e., $x \in C$. So $A \cap C \neq \varnothing$. Let $a \in A \cap C$. Then $a \in A$ and $S\Gamma a \subseteq S\Gamma A \subseteq A$. So $a \cup S\Gamma a \subseteq A$ and $(a \cup S\Gamma a] \subseteq (A] = A$. Since $L(a) \subseteq A$, $C \subseteq L(a)$, and $S \setminus C \subset A$, we have $S = (S \setminus C) \cup C \subseteq A \cup L(a) \subseteq A \subseteq S$. Hence $S = A$. Therefore, $S \setminus C$ is a maximal proper left Γ-ideal of S. □

Theorem 2.14 : *Let C be the union of all right bases of a po-*Γ*-semigroup S such that* $\varnothing \neq C \subset S$. *If S contains a maximal left* Γ*-ideal which contains every proper left* Γ*-ideal of S, denoted by* M^*, *then* $S \setminus C = M^*$ *if and only if*

(1) $|A| = 1$ *for every right base A of S;*

(2) one of the following conditions holds:

(2.1) $(S\Gamma A] = S$ *for every right base A of S;*

(2.2) S contains only one right base $A = \{a\}$ *with* $(S\Gamma a] \neq S$ *but* $a \notin (S\Gamma a]$.

Proof : Assume that $S \setminus C = M^*$. Then $S \setminus C$ is a maximal proper left Γ-ideal of S. By Theorem 2.13, $C \subseteq L(a)$ for all $a \in C$. We will show that $S \setminus C \subseteq L(a)$ for all $a \in C$. Suppose that $S \setminus C \nsubseteq L(a')$ for some $a' \in C$. Then $L(a') \subset S$, and $L(a')$ is a proper left Γ-ideal of S. So $L(a') \subseteq M^* = S \setminus C$, and hence $a' \in S \setminus C$. This is a contradiction. Then $S \setminus C \subseteq L(a)$ for all $a \in C$. By $S = (S \setminus C) \cup C \subseteq L(a) \subseteq S$ for all $a \in C$, it follows that $L(a) = S$ for all $a \in C$. Therefore, $\{a\}$ is a right base of S for all $a \in C$. Let A be a right base of S, and let $a, b \in A$. Suppose that $a \neq b$. Since $A \subseteq C$, $a \in C$, and so $S = L(a)$. Since $a \neq b$ and $b \in S = (a \cup S\Gamma a]$, then $b \in (S\Gamma a]$. By Lemma 2.3, $a = b$. This is a contradiction. Thus $a = b$ and $|A| = 1$. Therefore (1) holds. Next, we will show that (2.1) or (2.2) holds. Assume that (2.1) is false. Then there exists a right base $A = \{a\}$ of S such that $(S\Gamma a] \neq S$. If $a \in (S\Gamma a]$ and $(a] \subseteq ((S\Gamma a]] = (S\Gamma a]$ then $(S\Gamma a] = (a] \cup (S\Gamma a] = (a \cup S\Gamma a] = S$. This is a contradiction. Thus $a \notin (S\Gamma a]$. Let $A_1 = \{a_1\}$ be a right base of S such that $(S\Gamma a_1] \neq S$ and $a_1 \notin (S\Gamma a_1]$. Assume that $A \neq A_1$. We will show that $\{a_1\} = (a_1]$. Suppose that $b \in S \setminus A_1$ such that $b \leq a_1$ where $a_1 \in A_1$. Since $b \leq a_1$, by Lemma 2.5, $b \leq_L a_1$. Then $b \in L(b) \subseteq L(a_1) = L(A_1)$. So $S \setminus A_1 \subseteq L(A_1)$. We have $A_1 \subseteq L(A_1)$, then $S \setminus L(A_1) \subseteq S \setminus A_1 \subseteq L(A_1)$. This is a contradiction. Thus $\{a_1\} = (a_1]$. Since $a \in S = (a_1 \cup S\Gamma a_1] = (a_1] \cup (S\Gamma a_1]$, so $a \in (S\Gamma a_1]$. We obtain $A_1 \subseteq C$, $S \setminus C \subseteq S \setminus A_1 = (a_1 \cup S\Gamma a_1] \setminus \{a_1\} = (a_1] \cup (S\Gamma a_1] \setminus \{a_1\} = (S\Gamma a_1] \subset S$. So $S \setminus C \subseteq (S\Gamma a_1]$. Since $a \in C$, $a \notin S \setminus C$

and $a \in (S\Gamma a_1]$, we have $S \setminus C \subset (S\Gamma a_1]$. This contradicts to the maximality $S \setminus C$. Thus $A = A_1$. Therefore (2.2) holds.

Conversely, assume that (1) and (2.1) hold. We will show that $S \setminus C = M^*$. Since $\emptyset \neq S \setminus C \subset S$, by Theorem 2.12, $S \setminus C$ is a proper left Γ-ideal of S. Let B be a left Γ-ideal of S such that $S \setminus C \subseteq B \subseteq S$. Suppose that $S \setminus C \neq B$. Then $S \setminus C \subset B$, we have $x \in B$ and $x \notin S \setminus C$, i.e., $x \in C$. So $B \cap C \neq \emptyset$. Let $a \in B \cap C$. Then $a \in A$ for some a right base A of S. By (1), $A = \{a\}$. By (2.1), $(S\Gamma a] = S$. Since $a \in B$, we have $(S\Gamma a] \subseteq (S\Gamma B] \subseteq (B]$. So $S = (S\Gamma a] \subseteq (B] = B \subseteq S$. Thus $B = S$. Hence $S \setminus C$ is a maximal proper left Γ-ideal of S. Next, let M be a proper left Γ-ideal of S. Suppose $M \nsubseteq S \setminus C$. Then there exists $a \in M$ and $a \notin S \setminus C$. We obtain $S\Gamma a \subseteq S\Gamma M$, and so $(S\Gamma a] \subseteq (S\Gamma M] \subseteq (M] = M \subset S$. Since $a \notin S \setminus C$, $a \in C$, so $a \in A$ for some a right base A of S. By (1), $A = \{a\}$. By (2.1), $(S\Gamma a] = S$. This is a contradiction. Hence $M \subseteq S \setminus C$. Therefore $S \setminus C = M^*$.

Finally, we assume that (1) and (2.2) hold. By (1), every right base of S has only one element. By (2.2), S contains only one right base $A = \{a\}$ with $(S\Gamma a] \neq S$ but $a \notin (S\Gamma a]$. We have $C = A = \{a\}$. Thus $S \setminus A = S \setminus C \neq \emptyset$. By Theorem 2.12, $S \setminus C$ is a proper left Γ-ideal of S. Let B be a left Γ-ideal of S such that $S \setminus C \subseteq B \subseteq S$. Suppose that $S \setminus C \neq B$. Then $S \setminus C \subset B$. So $C \cap B \neq \emptyset$. Let $a \in C \cap B$. Then $S = (a \cup S\Gamma a] \subseteq (B \cup S\Gamma B] \subseteq (B] = B \subseteq S$. We have $S = B$. Hence $S \setminus C$ is a maximal proper left Γ-ideal of S. Let M be a proper left Γ-ideal of S. If $M \not\subseteq S \setminus C$, then exists $a \in M$ such that $a \notin S \setminus C$. So $S = (a \cup S\Gamma a] \subseteq (M \cup S\Gamma M] \subseteq (M] = M \subseteq S$. Thus $S = M$. This is a contradiction. Hence $M \subseteq S \setminus C$. Therefore, $S \setminus C = M^*$.

3. Conclusion and Discussion

In this paper, we prove that a non-empty subset A of a po-Γ-semigroup S is a right base of S if and only if A satisfies the following two conditions for any $x \in S$ there exists $a \in A$ such that $x \leq_L a$, and for any two distinct elements $a, b \in A$ neither $a \leq_L b$ nor $b \leq_L a$. Also, we prove the right bases of a po-Γ-semigroup S have the same cardinality. Finally, let C be the union of all right bases of a po-Γ-semigroup S, we prove that if $S \setminus C$ is non-empty, then $S \setminus C$ is a left Γ-ideal of S.

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