International Journal of Mathematics and
Computer Science, **17**(2022), no. 1, 135–142

A note of 2-distance balancing numbers

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(Received June 22, 2021, Accepted July 23, 2021)

Abstract

In this paper, we define and examine the concept of 2-distance balancing numbers. Moreover, we investigate some properties of those numbers and their recurrence relation. Furthermore, we provide the generating functions and Binet formula for 2-distance balancing numbers.

1 Introduction and Preliminaries

In [1], Behere and Panda introduced and investigated the concept of balancing numbers and balancers. A positive integer $n > 1$ is called a *balancing number* if

$$1 + 2 + 3 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r), \qquad (1.1)$$

Key words and phrases: 2-distance balancing numbers, generating functions, perfect square.
AMS (MOS) Subject Classifications: 05A15, 65Q30.
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ISSN 1814-0432, 2022, http://ijmcs.future-in-tech.net

for some $r \in \mathbb{Z}^+$ and r is called the *balancer* corresponding to the balancing number n; for instance, the numbers $6, 35$ and 204 are balancing numbers with balancers $2, 14$ and 84, respectively. In addition, they gave the generating functions and the recurrence relation for balancing numbers. Later, Panda and Ray [5] defined the cobalancing numbers and the cobalancers as follows:
A positive integer n is called a *cobalancing number* if

$$1 + 2 + 3 + \cdots + n = (n+1) + (n+2) + \cdots + (n+r), \tag{1.2}$$

for some $r \in \mathbb{Z}^+$ and r is called the *cobalancer* corresponding to cobalancing number n. For example, the number $2, 14$ and 84 are cobalancing numbers with cobalancers $1, 6$ and 35, respectively. Panda and Ray investigated some generating functions and recurrence relations for cobalancing numbers. Moreover, they gave the relations among balancing numbers, cobalancing numbers, balancers and cobalancers. The balancing numbers, cobalancing numbers and their generalizations are widely studied by several researchers. Liptai focused on studying Fibonacci balancing numbers and Lucas balancing numbers in [2] and [3], respectively. Later, Olajos [4] gave some interesting properties and results on balancing numbers, cobalancing numbers and many types of generalized balancing numbers. Ray [6] extended the concept of balancing numbers to k-balancing numbers and presented that the balancing polynomials are the natural extension of k-balancing numbers.

In this paper, we modify the definition of balancing numbers and cobalancing numbers to 2-distance balancing numbers in the same manner. A positive integer $n > 2$ is called a 2-*distance balancing number* if

$$1 + 2 + 3 + \cdots + (n-2) = (n+2) + (n+3) + \cdots + (n+r), \tag{1.3}$$

for some $r \in \mathbb{Z}^+$ and r is called the 2-*distance balancer* corresponding to 2-distance balancing number n. For example, the number $8, 47$ and 274 are 2-distance balancing numbers with 2-distance balancers $3, 19$ and 113, respectively. The purpose of this paper, is to introduce and examine the 2-distance balancing numbers. Moreover, we investigate some properties of those numbers and their recurrence relation. Furthermore, we provide the generating functions and Binet formula for 2-distance balancing numbers.

2 Generating function for 2-distance balancing numbers

In this section, we provide some basic properties of 2-distance balancing numbers and their generating functions.

Proposition 2.1. *A positive integer n is a 2-distance balancing number with 2-distance balancer r if and only if*

$$n^2+2=\frac{(n+r)(n+r+1)}{2} \tag{2.1}$$

and

$$r=\frac{-(2n+1)\pm\sqrt{8n^2+17}}{2} \tag{2.2}$$

Proof. Let n be a 2-distance balancing number with 2-distance balancer r. We obtain

$$\begin{aligned} 1+2+\cdots+(n-2) &= (n+2)+(n+3)+\cdots+(n+r)\\ 2[1+2+\cdots+(n-2)]+(n-1)+n+n+1 &= 1+2+\cdots+(n+r)\\ (n-1)(n-2)+3n &= \frac{(n+r)(n+r+1)}{2}\\ n^2+2 &= \frac{(n+r)(n+r+1)}{2} \end{aligned}$$

Then equation (2.1) holds and we have $r=\dfrac{-(2n+1)\pm\sqrt{8n^2+17}}{2}$. Conversely, if $n^2+2=\dfrac{(n+r)(n+r+1)}{2}$, then it is easy to see that n is a 2-distance balancing number with the 2-distance balancer r. □

By Proposition 2.1, we have

Proposition 2.2. *A positive integer n is a 2-distance balancing number if and only if n^2+2 is a triangular number and n is a 2-distance balancing number if and only if $8n^2+17$ is a perfect square.*

In the remainder of this section, we introduce some functions that generate 2-distance balancing numbers. For any 2-distance balancing number x, we consider the following functions :

$$f(x)=3x+\sqrt{8x^2+17} \tag{2.3}$$

$$g(x) = 17x + 6\sqrt{8x^2 + 17} \tag{2.4}$$

We show that the above functions generate 2-distance balancing numbers as in the following theorem.

Theorem 2.3. *For any 2-distance balancing number x, $f(x)$ and $g(x)$ are also 2-distance balancing numbers.*

Proof. Since $8x^2 + 17$ is a perfect square, it follows that

$$8(f(x))^2 + 17 = (8x + 3\sqrt{8x^2 + 17})^2$$

is also a perfect square. Hence $f(x)$ is a 2-distance balancing number. In the same way, since $f(f(x)) = g(x)$, this implies that $g(x)$ is also a 2-distance balancing number. $\square$

Example 2.1. We know that $x = 8$ is a 2-distance balancing number with the 2-distance balancer 3 because

$$1 + 2 + 3 + 4 + 5 + (8 - 2) = 21 = (8 + 2) + (8 + 3).$$

We obtain $f(8) = 47$ and $g(8) = 274$ are 2-distance balancing numbers.

Theorem 2.4. *If x is any 2-distance balancing number, then there is no 2-distance balancing number y such that $x < y < 3x + \sqrt{8x^2 + 17}$.*

Proof. The function $f : [0, \infty) \to [17, \infty)$, defined by

$$f(x) = 3x + \sqrt{8x^2 + 17}$$

is strictly increasing since

$$f'(x) = 3 + \frac{8x}{\sqrt{8x^2 + 17}} > 0.$$

We have that f is bijective and $f(x) > x$ for all $x \geq 0$. Hence, f^{-1} exists and is also strictly increasing with $f^{-1}(x) < x$. Let $u = f^{-1}(x)$. Then $f(u) = x$ and $u = 3x \pm \sqrt{8x^2 + 17}$. Since $u < x$, we obtain that $u = 3x - \sqrt{8x^2 + 17}$. Since $8(f^{-1}(x))^2 + 17 = (8x - 3\sqrt{8x^2 + 17})^2$ is a perfect square, this implies that $f^{-1}(x)$ is also a 2-distance balancing number.

Let H_i be the hypothesis that there is no 2-distance balancing number between b_{i-1} and b_i, for an integer $i \geq 1$. Assume that H_n is false for some n. Thus, there exists a 2-distance balancing number y such that $b_{n-1} < y < b_n$ and it follows that $b_{n-2} < f^{-1}(y) < b_{n-1}$. Finally, this would imply that there exists a 2-distance balancing number b between b_0 and b_1, which is false. Therefore, H_n is true for any positive integer n. $\square$

The next corollary follows by Theorem 2.4.

Corollary 2.5. *If x is any 2-balancing number, then its previous 2-distance balancing number is $3x - \sqrt{8x^2 + 17}$.*

3 Recurrence relations for 2-distance balancing numbers

For $n = 1, 2, 3, \ldots$ let b_n be the nth 2-distance balancing number. Since $8(1)^2 + 17 = 25$ is perfect square, we accept that 1 is a 2-distance balancing number and we set $b_0 = 1$. Since the next 2-distance balancing numbers are $8, 47, \ldots$ we also set $b_1 = 8$, $b_2 = 47$ and so on.

From Corollary 2.5, we suggest that

$$b_{n+1} = 3b_n + \sqrt{8b_n^2 + 17} \tag{3.1}$$

$$b_{n-1} = 3b_n - \sqrt{8b_n^2 + 17} \tag{3.2}$$

By adding the above two equations, we get the following recurrence relation:

$$b_{n+1} = 6b_n - b_{n-1}, n \geq 1 \tag{3.3}$$

Theorem 3.1. *For any positive integer $n > 1$, the following are true:*

(*a*) $b_n^2 = b_{n-1}b_{n+1} + 17$

(*b*) $(b_{2n} - 4)(b_{2n} + 4) = 1 + b_{2n-1}b_{2n+1}$

(*c*) $b_n = b_m b_{n-m} - b_{m-1}b_{n-m-1} - 2b_{n-1}$*; $m < n$.*

Proof. From equation (3.3), we have

$$\frac{b_{n+1} + b_{n-1}}{b_n} = 6. \tag{3.4}$$

Replacing n by $n - 1$, we obtain

$$\frac{b_n + b_{n-2}}{b_{n-1}} = 6 \tag{3.5}$$

which implies that

$$\frac{b_n + b_{n-1}}{b_n} = \frac{b_n + b_{n-2}}{b_{n-1}}. \tag{3.6}$$

Thus,

$$b_n^2 - b_{n+1}b_{n-1} = b_{n-1}^2 - b_n b_{n-2} \tag{3.7}$$

Now, iterating recursively, we obtain

$$b_{n-1}^2 - b_n b_{n-2} = b_1^2 - b_0 b_2 = (8)^2 - (1)(47) = 17 \tag{3.8}$$

Hence, $b_n^2 = b_{n+1}b_{n-1} + 17$. Therefore, the proof of (a) is complete.
From part (a) we prove (b) by replacing n by $2n$. Thus, we have

$$b_{2n}^2 - 16 = 1 - b_{2n+1}b_{2n-1}.$$

To prove (c), we use the mathematical induction on $n > 1$. In case $m = 1$, we have

$$b_1 b_{n-1} - b_0 b_{n-2} - 2b_{n-1} = 8b_{n-1} - b_{n-2} - 2b_{n-1} = 6b_{n-1} - b_{n-2} = b_n.$$

Assume that it is true for $m = k$. Consider

$$\begin{aligned} b_{k+1}b_{n-k-1} - b_k b_{n-k-2} - 2b_{n-1} &= (6b_k - b_{k-1})b_{n-k-1} - b_k b_{n-k-2} - 2b_{n-1} \\ &= b_k(6b_{n-k-1} - b_{n-k-2}) - b_{k-1}b_{n-k-1} - 2b_{n-1} \\ &= b_k b_{n-k} - b_{k-1}b_{n-k-1} - 2b_{n-1} \\ &= b_n. \end{aligned}$$

Thus, it is true for $m = k+1$. Consequently, the proof of (c) is complete. □

4 Generating function and Binet formula for 2-distance balancing numbers

In the previous section, we obtained some recurrence relations for the sequence of 2-distance balancing numbers. In this section, we consider the generating function and the Binet formula for 2-distance balancing numbers. Recall that the generating function for a sequence $\{x_n\}$ of real numbers is defined by

$$g(s) = \sum_{n=0}^{\infty} x_n s^n.$$

Theorem 4.1. *The generating function for the sequence of 2-distance balancing numbers* $\{b_n\}_{n=1}^{\infty}$ *is*

$$F(s) = \frac{8s - s^2}{1 - 6s + s^2}. \tag{4.1}$$

Proof. From (3.3), for $n = 1, 2, ...$ we have $b_{n+2} - 6b_{n+1} + b_n = 0$. Multiplying both sides by $s^n + 2$ and summing, we obtain

$$\sum_{n=1}^{\infty} b_{n+2}s^{n+2} - 6s\sum_{n=1}^{\infty} b_{n+1}s^{n+1} + s^2\sum_{n=1}^{\infty} b_n s^n = 0 \tag{4.2}$$

which in terms of $F(s)$ can be expressed as

$$(F(s) - (8s + 47s^2)) - 6s(F(s) - 8s) + s^2F(s) = 0. \tag{4.3}$$

Therefore, we get

$$F(s) = \frac{8s - s^2}{1 - 6s + s^2}. \tag{4.4}$$

□

Theorem 4.2. *If b_n is the n^{th} 2-distance balancing numbers, then*

$$b_n = \frac{(5 + \sqrt{8})\alpha^n - (5 - \sqrt{8})\beta^n}{\alpha - \beta}, \tag{4.5}$$

where $\alpha = 3 + \sqrt{8}$ and $\beta = 3 - \sqrt{8}$ are roots of a characteristic equation $x^2 - 6x + 1 = 0$ for the recurrence relation (3.3).

Proof. Since $\alpha = 3 + \sqrt{8}$ and $\beta = 3 - \sqrt{8}$ are real numbers, we have

$$b_n = A\alpha^n + B\beta^n \tag{4.6}$$

where A and B are determined from the value of b_0 and b_1. Substituting $b_0 = 1$ and $b_1 = 8$ into (4.6), we obtain

$$A + B = 1 \tag{4.7}$$

$$A\alpha + B\beta = 8. \tag{4.8}$$

Solving above equations for A and B, we get $A = \dfrac{5 + \sqrt{8}}{\alpha - \beta}$ and $B = \dfrac{5 - \sqrt{8}}{\alpha - \beta}$. Substituting these values into (4.6), we obtain

$$b_n = \frac{(5 + \sqrt{8})\alpha^n - (5 - \sqrt{8})\beta^n}{\alpha - \beta}.$$

We call (4.5) the Binet formula for 2-distance balancing numbers. □

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